OPTIMIZATION IN CARTESIAN CLOSED CATEGORIES, LAGRANGE'S METHOD OF MULTIPLIERS AND APPLICATIONS

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DECLARATION AND RECOMMENDATION

DECLARATION

This thesis is my original work and has not been submitted in part or whole for an award in any university.

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RECOMMENDATION

This thesis has been submitted for examination with our approval as university supervisors.

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DEDICATION

To

My late father Meshack Jason Jagongo

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ABSTRACT

A category is defined as an algebraic structure that has objects that are linked by morphisms. Categories were created as a foundation of mathematics and as a way of relating algebraic structures and systems of topological spaces. Any foundation of mathematics must include algebra, topology, and analysis. Algebra and topology have been studied extensively in category theory but not the analysis. This is partly due to the algebraic nature of category theory and the fact that the axiom of choice is not used in category theory. However, with the introduction of infinitesimals, it has been possible to study synthetic differentiation that is consistent with categories. It has been pointed out that, in order to treat mathematically the decisive abstract general relations of physics, it is necessary that the mathematical world picture involves a Cartesian closed category of smooth morphisms between smooth spaces. Algebra and topology have been studied in Cartesian closed categories but optimization has not so far been considered. This study aimed at defining a cone, derivative, extremal object and then used these definitions to obtain optimization results using the Lagrange method of multipliers in Cartesian closed categories. The study also provides a discussion of the various areas that the results can be applied such as building spreadsheet application, neuroscience, cognitive neural network architectures and program optimizations.

TABLE OF CONTENTS

LIST OF NOTATIONS

$\rightarrow$ Map

$\varnothing$ Empty set

$\in$ Member/Element of

$\forall$ For all/every

$\cap$ Intersection

$\cup$ Union

$\exists$ There exists

$\subset$ Subset

$\supset$ Superset

CHAPTER ONE

INTRODUCTION

1.1 Background Information

In the context of category theory, a category is said to be Cartesian closed if any morphism defined on a product of two objects can be naturally identified with a morphism defined on one of the factors (Hirschowitz, 2013). Cartesian closed categories are very important in the study of mathematical logic and the theory of programming in that their internal language is the simply typed lambda calculus.

Cartesian closed categories are sometimes generalized by closed monoidal categories. A closed monoidal category is a context where there is a possibility to form both tensor products and to form mapping objects. A classic example is the category of sets in which case the tensor products of the sets A and B is the usual Cartesian product $A \times B$ and the mapping object B^A is the set of functions from A to B.

Optimization is one of the key components of mathematical modelling of real world problems and the solution method should provide an accurate and essential description and validation of the mathematical model. It is important to note that optimization problems are always encountered frequently in engineering and in sciences and have widespread practical applications. For instance, it is possible to optimize an approximately chosen cost functional subject to some constraints in economics. The constraints are always either in the form of equality constraints or inequality constraints.

The study of optimization in categories seem problematic due to the fact that it is algebraic and yet most optimization problems are studied using classical analysis. Since Lawvere (1963) and Kock(1981) innovated the use of infinitesimals, it has been possible to study some parts of analysis in such toposes. Extrema properties of a complex variable do not exist since most of the extrema problems involve the order properties of the real line. Sukhinin (1982) introduced the concept of extrema in spaces without norm that is applicable even to functions of complex variables. The idea has been adopted to obtain optimization results in topological spaces without norm. Otieno et al (2013) also adopted the idea and obtained optimization results in ordered topological modules. This study also seeks to adopt the idea to obtain optimization results in Cartesian closed categories.

The use of ordered vector spaces and cones in mathematics is very important and began around 1950 through the efforts of Dutch, Russian, German and Japanese mathematicians (Otieno et al, 2013). The importance of cones in the areas of optimization and fixed point theory cannot go unnoticed as it is through cones that optimization results are obtained. Cones are usially employed to solve various optimization problems and it is for this reason that the theory of ordered vector spaces is a tool for solving various applied problems in areas such as physical science, engineering, social science and econometrics.

Cones are vital in the study of optimization in category theory. If V is a pre-ordered vector space, the subset $C = \{x \in V : x \geq 0\}$ is a convex cone also known as a positive cone. In a nutshell, a cone is said to be a convex cone if a straight line can be drawn inside the cone. This concept is useful for any vector space that allows the use of positive scalars such as spaces over real numbers, rational numbers or even algebraic numbers. By the definition of a convex cone, it follows that any linear subspace of V including the empty set, the trivial subspace $\{\varnothing\}$ and the set V itself are convex cones. In the same vector space, the intersection of two convex cones is a convex cone but the union of two convex cones need not be a convex cone (De et al, 2012). Under arbitrary linear maps, the family of convex cones is closed. Moreover, if C is a convex cone its opposite -C is also convex and $C \cap -C$ is the largest linear subspace contained in the convex cone.

The study of category theory and especially Cartesian closed category can be very useful and it is not used in the same manner some aspects of mathematics are used (Voevodsky, 2010). For instance, optimization theory can be applied by noticing that a particular problem has a certain form and therefore a certain algorithm will converge to a solution. Applications of category theory are usually more subtle and it has been seen that category theory can be used in various applications. Category theory have been used to model some applications such as LINQ(Language Integrated Query) as well as being used to guide mathematical modelling and software development (Wachsmuth, 2013). It is possible to spot inconsistencies and errors in category theory similar to the way dimensional analysis does in engineering, or type checking in software development.

1.2 Statement of the Problem

Problems involving optimization have not been studied in Cartesian closed categories. This study aimed at defining a cone, derivative, extremal object and then used these definitions to obtain optimization results using the Lagrange method of multipliers in Cartesian closed categories. The study also provides a discussion of the various areas that the results can be applied such as building spreadsheet application, neuroscience, cognitive neutral network architectures and program optimizations. The results obtained in this study are vital as they aid in computerization of mathematics.

1.3 Objectives

1.3.1 General Objective

The study aimed at defining a cone, a derivative and then used these definitions to obtain optimization results in Cartesian closed categories by lagrange's method of multipliers.

1.3.2 Specific Objectives

1. To define and study the properties of a derivative in a Cartesian closed category.
2. To define and study the properties of a cone in a Cartesian closed category.
3. To prove the Lagrange method of multipliers in the Cartesian closed categories.
4. To give some application of optimization.

1.4 Justification

The methods of optimization are always designed in such a way that they can provide the best values for system design. There are many approaches to optimization depending on the space that is being used. This study seeks to prove optimization in Cartesian closed categories as to unite the different approaches in various spaces. This study is vital since, for foundations of mathematics to encompass physics, it always must have "the principle of least action" or optimization in ordinary mathematical language. This is because Physics is the cornerstone in the study of self-organisation. The results of this study are vital since they will aid in various applications such as building spreadsheet application, neuroscience, cognitive neutral network architectures and program optimizations.

CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction

A category is defined to be an algebraic structure that has objects that are linked by morphisms (Marquis, 2010). There are two basic properties that are associated with a category, the ability to compose the morphisms associatively and the existence of an identity morphism for each object in the category. Category theory has in the past represented a huge change in the way most mathematicians thought about mathematics, leaving the set theoretic nature of mathematics behind and bringing up the importance of arrows between the objects rather than the objects themselves (Nel, 1976). Generally, category theory brought up the aspect of diagrammatic mathematics in which case proofs are constructed in a diagrammatic manner.

Almost all the branches of modern mathematics can be represented using categories and in doing so deep insights and similarities are revealed between different areas of mathematics (Awodey, 2010). The disciplines of applied mathematics and computer science have advanced from the applications of constrained optimization; and specifically the lagrange's multiplier technique. This approach as indicated by Nesterov (2018) focuses on the refinements in the use of the Lagrange's method of multiplier which entails application in partial differential equations and other mathematical theories. The method is applied in three forms for the inequality, equality and non-distinguishable optimization problems. Nesterov (2018) suggests that the discussions emphasize on the procedures of approximation for ill-conditioned and non-differentiable optimization issues. The problems include the duality framework, exact minimization in the multiplier methods; and the method of quadratic penalty functions. The penalty approach was further examined inclusive of the non-distinguishable penalty functions and algorithm linearization as well as differentiable penalty functions and global and local lagrange methods of convergence.

Category theory was first introduced by Eilenberg and MacLane (1945) in their article titled "General theory of natural equivalences." It was in the 1950's that various mathematicians including Grothendiek utilized category theory quite successfully in the field of algebraic geometry (Vistoli, 2004). In particular, Grothendiek utilized category theory to develop Grothendiek toposes. It was applied to logic by Lawvere (1960) by creating categorical logic and further introduced Lawvere theories as a category-theoretic way to describe finitary algebraic theories. It is believed that the radical change in category theory originated in philosophy and was later absorbed in mathematics and this is the reason it is widely used in logic. In the recent past, category theory has been widely used to help computerise mathematics by trying to develop programs to proof mathematical theorems. Various mathematicians have been doing research so as to develop programs to prove the theorems in mathematics. Category theory has been widely used in the development of such programs. Computer science applies the concept of Cartesian closed categories by a process known as currying, which has led to the realization of the lambda calculus that is interpreted in various categories. It is clear from past research that there is a strong isomorphism among Cartesian closed categories, typed lambda calculus, and intuitionist logic.

2.2 Cartesian Closed Categories

In category theory, a category is said to be Cartesian closed if any morphism that is defined on a product of two objects can be naturally identified on one of the factors (Escardó, Lawson, & Simpson, 2004). These categories are seen to be very important in the theory of programming and in mathematical logic, in that their internal language is basically the typed lambda calculus. The Cartesian closed categories are generalised by closed monoidal categories whose internal language, linear type systems, are suitable for both classical and quantum computation (Höhle, 1991). A closed monoidal category is defined as a closed category M if it is equipped by finite product results with respect to the monoidal structure of its Cartesian. Its internal $\hom[S, X]$ in Cartesian closed categories is normally referred to as exponentiation which is represented by X^S. The functor of the cartesian closed categories between the Cartesian closed categories C and D is product-preserving as well as exponential preserving which forms a canonical diagram that corresponds to its composite thus implying isomorphism.

Some of the examples of cartesian closed categories include the category set of all sets with morphisms being the functions. The Cartesian product of X and Y is the product $X \times Y$, and $Z^Y = \{f : Y \to Z\}$. The following fact provides an expression for the adjointness: the curried function $g : X \to Z^Y$ which is defined by $g(x)(y) = f(x, y)\ \forall x \in X, y \in Y$ is used to naturally identify the function $f : X \times Y \to Z$. Another category that is Cartesian closed for a similar reason is the category of finite sets with morphisms being the functions. In this sense it is possible to model the typed lambda calculus by use of the Cartesian closed categories. Since it has finite products there is an analysis map that corresponds to the map of identity undergoing bijection and with application of naturality, the map is translated to its composite.

Cartesian closed categories (CCC) are generally easy to work with in algebraic topology. Neither the category of smooth manifolds containing smooth maps nor the category of topological spaces with continuous maps is Cartesian closed. For this reason, there are substitute categories that have been considered. The category of Hausdorff spaces that have been completely generated is Cartesian closed as is the category of Frolicher spaces (Nishimura, 2010). In order theory, complete partial orders have a natural topology that is known as the Scott topology whose continuous maps do form a Cartesian closed category. The objects in this category are the complete partial orders while the morphisms are the Scott continuous maps. In the Scott topology, both currying and apply are continuous functions and currying together with apply provide the right adjoints. A category that has a zero object is also said to be Cartesian closed if and only if it is equivalent to a category with one object and one identity morphism. This concept implies that it preserves all the co-limits; specifically, a CCC is known as a distributive category if it results in finite co-products. Another important factor to consider is the fact that CCCs' internallogic is the typed λ-calculus.

Cartesian closed categories have a lot of applications especially in computer science (Curry et. al., 1972). A function of two variables (a morphism $f : X \times Y \to Z$) can always be represented as a function of one variable (the morphism $\lambda f : X \to Z^Y$). In computer science applications, this is known as currying and it has led to the realization that the interpretation of simply-typed lambda calculus can be done in any Cartesian closed category (Mitchelle, 1996; Simpson, 1995). The Curry-Howard-Lambek correspondence provides a deep isomorphism between simply-typed lambda calculus, intuitionistic logic and Cartesian closed categories (Chakraborty, 2011). It is important to note that there has been a proposal that the Cartesian closed category, the topoi be a general setting for mathematics instead of the traditional set theory.

It is important to note that there is a tendency to distinguish between data and methods in imperative programming whereas in functional programming for example, they are identical. A Cartesian closed category makes this identification of hom-sets (methods) correspond to certain objects (data). In particular, the category is closed if it has a binary operation on objects $(X,Y) \mapsto Y^X$ such that it is actually the data version of methods:

$\forall X,Y,Z \bullet (Z \oplus X \to Y) \cong (Z \to Y^X)$ natural $Z,Y.$ from the Cartesian part, any finite record can be formed in the programming sense. That is, if there are n records of interest $R_1,...,R_n$, they can be amalgamated into one record $\prod_{i \in 1..n} R_i$ and that this is their product means that any function/method to it is precisely n methods to the individual factors.

Theorem 1 presents the necessary conditions that a category must fulfil in order to be Cartesian closed:

2.2.1 Theorem 1

The category C is said to be Cartesian closed if and only if it satisfies the following three properties:

i) It has a terminal object.

ii) Any two objects $X, Y \in C$ have a product $X \times Y \in C$.

iii) Any two objects $Y, Z \in C$ have an exponential $Z^{Y} \in C$.

Because of the natural associativity of the categorical product and also since the empty product in a category is the terminal object, properties (i) and (ii) can be combined to a single requirement that any family of objects of C admit a product in C. The finite family of objects can be empty. The property (iii) is equivalent to the fact that the functor $-\times Y$ denoted $-Y$, $\forall Y \in C$ (Escardó, Lawson, & Simpson, 2004).

2.3 Boundary and extrema

Given two real axes X and Y and a function $f : X \to Y$ that takes a closed bounded set $A \subset X$ to a closed bounded set $B \subset X$ then the problems of maxima and minima involves obtaining points in the set A which are mapped by f to either the maximum or the minimum points of B. If for instance, the closed bounded set is not the real line then there will be no maximum or minimum. However, it is possible to introduce the concept of extremal point as a point that is in A that is mapped by the function f to the boundary of B. This is done using locally convex topological vector spaces with bounded topology. This is done because classical calculus operation works up well upto the abstraction of the Banach spaces but not beyond. Further, developments have shown that classical calculus can still work well in topological vector spaces more general than Banach spaces provided the topology used is bounded. Another good reason for this topological space is that it is amendable to category theoretic approach of study (Andreas and Peter, 1997). In cartesian closed category the concept of boundary and extrema can be defined using the concept of locally convex topological vector spaces. Although there are calculation tests used to determine the values that satisfy the lagrange's multiplier for the local maxima and minima as well as the points of inflection. The most appropriate way is to undertake examination of the function's contour plots.

2.4 Lagrange method of multipliers

The Lagrange method of multipliers is vital while dealing with optimization problems that are constrained and is also related to many other results which are important. There are many different ways in which the fundamental result can be obtained. One of the ways in which the fundamental result can be obtain is through the use of the variational approach which provides a deep understanding on the nature of the rule of Lagrange multiplier (Borwein and Zhu, 2005). Gale (1967), provides a penetrating explanation on the economic meaning of the Lagrange multiplier in the convex case. It is important to note that the Lagrange method of multiplier is mainly applicable in the convex case (Nesterov, 2018).

Consider maximizing the output of an economy with resource constraints. Then the optimal output is a function of the level of resources. It turn out that if the derivative of this function exists, it is exactly the Lagrange multiplier for the constrained optimization problem. A Lagrange multiplier then reflects the marginal gain of the output function with respect to the vector of resource constraints. Following this observation, if the resource utilization is penalized with a (vector) Lagrange multiplier the the optimization problem that has been constrained can be converted to an optimization problem that is unconstrained. One cannot emphasize enough the importance of this insight.

In general, however, an optimal value function for a contrained optimization problem is neither convex nor smooth. This explains the reason behind the fact that this view was not prevalent before the systematic development of nonsmooth and variational analysis. This systematic development of convex and nonsmooth analysis during the 1950s through 1970s, respectively, provided tools that are suitable for the proper analysis of the Lagrange multiplier (Nesterov, 2018). Gale (1989) himself provided a rigorous proof of the fact that for convex problems that are well behaved the subdifferential of the optimal value function exactly characterizes the set of all Lagrange multipliers.

Subsequently, many researchers have derived versions of the Lagrange multiplier theorems with different ranges of applicability using other generalized derivative concepts (Maugeri & Puglisi, 2014). It is a methodology that aids in deriving the values of the maxima and minima of a particular function subject to constraints of equality. In short, it is a powerful tool utilized in the providing solution to such class of mathematical problems without the necessity of explicitly resolving the conditions of constraint as well as uses them in elimination of extra variables. According to Almeida and Torres (2009), the function provides a description for the dynamic system state in form of time derivatives and position coordinates, which is equivalent to the distinction between kinetic and potential energy.

Despite the extensive literature on the various Lagrange multiplier rules, there are some finer points which are worth mentioning. First, the Lagrange multipliers are intrinsically related to the derivative or to derivative like properties of the optimal value function. This is already well explained from the economic explanation of the Lagrange multiplier rule in Gale's (1989) paper. Gale (1989) focuses on the convex case but the essential relationship extends to the Lagrange multiplier rules that rely on other generalized derivatives. Second, in a Lagrange multiplier rule a complementary slackness condition holds when the optimal solution exists. Nonetheless, without a prior existence of an optimal solution, a Lagrange multiplier rule involving only the optimal value function still holds and is often useful (Benoist, Borwein and Popovici, 2002). Third, the form of a Lagrange multiplier rule is always dictated by the properties of the optimal value function and by the choice of the generalized derivative. In many developments, sufficient conditions for ensuring the existence of such generalized derivatives are not always clearly disentangled from what was necessary to derive the Lagrange multiplier rule itself. Finally, computing the Lagrange multiplier often relies on the decoupling information in terms of each individual constraint. Sufficient conditions are often needed for this purpose and they are not always clearly limned.

The Lagrange method of multiplier used to be viewed as auxiliary variables introduced in a problem of constrained minimization in order to write first-order optimality conditions formally as a system of equations (Shapiro, 1997). Modern applications which have put more emphasis on numerical methods and more complicated side conditions than equations have demanded a deeper understanding of the concept and how it fits into a larger theoretical picture (Bertsekas, 2014).

Finding maxima and minima is among the most common problems in Calculus, but it is always difficult to obtain a closed form of the function that is being extremized. Consider the optimization problem: maximize $f(x,y)$ subject to $g(x,y)$. We introduce a new variable λ called the Lagrange function and is defined by:

$\omega(x,y,\lambda) = f(x,y) + \lambda\{g(x,y) - c\}$ Where λ term may be added or subtracted. It is important to note that not all stationary points yield a solution to the original problem. As stated by Vapnyarskii (2001), this method yields a necessary condition for optimality in constrained problems. The rule of Lagrange multipliers in locally convex spaces and the concept of extrema can be extended to Cartesian closed categories by modifying the Sukhinin's (1982) method to rings (Cartesian closed categories). Sukhinin (1982) studied Lagrange multipliers in vector spaces with bounded topology and it is this idea that we intend to generalize to rings (Cartesian closed categories). Sukhinin used the idea of cones in linear topological spaces without norms in order to obtain results similar to those obtained in classical analysis.

The problem of optimization has been studied using the concept of infinitesimals in spaces without norm. Otieno et al, (2013) used this concept and obtained the results shown in Theorem 2. It is these results that we use in this study to obtain optimization results in Cartesian closed categories.

2.4.1 Theorem 2

Let xo∈S ⊂ X, let K1 be a cone in X with vertex at zero, and suppose that the map f: S →Y is such that ∀h∈K1,∀U∃B∃δ∀t∈(0,δ)∃x∈t(U∩B):[xo+ th + x∈S]∧[f(xo+ th + x) = f(xo)] and f (xo) = 0.Further, let Z be an ordered topological vector space, suppose that the map F: S → Z is β differentiable relative to K1 at the point xo. If xo is a point of conditional β minimum of the map F relative to K1 and under the condition f (xo) = 0 then $F'(x_o)(h) \geq 0$ for h∈K1.

Sometimes, the theorem above can be given the form of the rule of Lagrange multipliers, as shown below;

Let Z be an ordered topological vector space, g be a continuous linear operator from X into Z, let A be a linear (not necessary continuous) operator from X into Y, K be a cone in X, and suppose that∀h∈ (Ker A) ∩K: g (h) ≥0 (Otieno, Sogomo, &Gichuki, 2013)

2.4.2 Proposition

Suppose ∀h∈ (Ker A) ∩K: g (h) ≥0 holds, g is linear, A is open $(AX = Y)$, and that $K = X$ then ∃ an operator $\lambda \in \omega(Y, Z)$ such that $(g + \lambda \circ A)x = 0$ for some $x \in X$.#

Indeed to optimize g subject to A, we introduce a new variable λ and define the lagrangian function $\omega = g + \lambda \circ A$, then the first order necessary conditions for a feasible point x_0 to be a locally optimal solution to the Lagrangian function ω becomes;

$$\left.\frac{\delta\omega}{\delta\lambda}\right|_{x_0} = 0 \text{ and} [\lambda, g(x_0)] = 0 \text{ for some } x_0 \in X$$

2.5 Typed lambda calculus and its relationship to computer foundations of mathematics

This calculus refers to the typed formalization that involves the use of the symbol (lambda) that stands for an unknown function abstraction. The calculus in consideration determines the actual nature of syntactic objects; mostly these are types regarding lambda terms. From an individual perspective, these calculi can be viewed as an improvement of the untyped calculus that was also lamda based. They are foundational theories that are contextually programming language hence forming the basis for functional coding like Haskell and ML (Li et al., 2017). These calculi play a vital role in the development of programming languages' type systems; in this case the aspect of typability normally focuses on desirability of the program properties for instance to prevent violation of memory access.

The typed calculus closely correlates to proof theory and mathematical logic through the concept of Curry-Howard of isomorphism and could be regarded as a specified language to a class in a category like for example the CCCs' (Cartesian closed categories) language is the simply-TLC. The concept of Lambda Calculus was first introduced in the year 1932 in a church system as the origin of logic. It is precisely the approach of studying philosophy centered computation regarding mathematical theories and logical formula. Lamda calculus depicts the deep connection between recursion and computing as well as the relation to mathematical induction (Kanzow & Steck, 2017).

The key roles of λ-calculus evident from past review according to CH correspondence include the basic mathematical foundation of functional, sequential, and high-order computational tendency and the fact that it is proofed' representation in constructive logic.

From the dual perspective of lambda calculus both as a programming language and proof in conjunction with the algebraic nature of the concept, it has been possible to perform a massive shift in technology between programming, the foundation of mathematics and logic. Majority of coding languages as well as typing fields cannot experience development without λ-calculus. It has contributed a great deal to the discipline of computational mathematics.

The concurrency theory is also one of the disciplines in computer science that has significantly affected the constitutional perspective according to Martin Burger. λ-calculus on its own is a non-concurrent language but has the spirit of algebra which permeates the development and definition of the prevailing process calculus. In short, the process algebras are λ-calculus descendants rather than being Turing machines or automata; hence the importance of importing λ-calculus to the concurrence theory. Besides this theory, the ICC (inherent computational complexity) is useless in CS outside use of coding languages as well as verification of software. λ-calculus is applied in partial differential equations since the unit forms part of mathematical formalism, relational algebra, type theory, and the higher order dimensional logic. The church had reasons for building the λ-calculus which vary according to the practitioner as follows: A convenient computing notation, unlike the typical Turing Machines; Provides a solid basis for mathematical manipulations to create a more complex programming language.

It is also rigorous equipment used to give natural semantics and coding languages.Functional programming technique is base its application in differential equations; a good example is the Lisp language (Bendkowski et al., 2017). The reduction systems are primarily designed to execute such functional languages. Since these programs have an expression denoted as E that represents both the input and algorithm. Certain rewrite principles guide the expression; the reduction procedure entails replacement of part N of E with N' subject to the rewrite rules as shown by the following schematic equation:

E|N|→ E|N'|, given that N →N' is by the rules.

In this sense the reduction process shall be redone until no parts are available for rewriting; the standard form of the expression constitutes of the provided functional program's output. These systems usually satisfy the property of Church-Rosser; it states that the expression's standard form realized is not dependent on the sub-terms by evaluation order. The basic operation of λ-calculus includes abstraction and application.

The application operation is whereby expressions L.W and LW imply that data L regarded as an algorithm implemented on data W assumed to be the input (Kanzow & Steck, 2017). This process can be viewed from two perspectives that include as an output or as a computation procedure. The computation view gets captured in the conversion notion (reduction) while the output part is featured in semantics notion. The other operation, abstraction, is whereby if forms the expression that contains, then the function →denoted by λ... These two operations work in conjunction according to the following intuitive expression (λ .)= [:=]; I which [:=] implies that substitutes (Palomba et al. 2017).

Church was precisely attempting to perform unification of the notations utilized in computing their mathematics. On realization that LC is equivalent to TM, they were used as a standard alternative to improve the accessibility of the programs. λ-calculus is treated like primitive instead of a dialect of TM since its semantics are termed as denotational for possessing intrinsic meanings. The meanings include the fact that they are church numbers, addition, recursion and multiplication functions. This aspect makes LC terms more aligned to the formal practice of mathematics hence the existence of several algorithms in LC form directly (Derpich & Sepulveda, 2017). The church's system utilized a logic that is type-free having unrestricted quantification but with no excluded middle law (Dershowitz, & Gurevich, 2008).

2.6 Hilbert 6th Problem

The sixth problem in the famous list of Hilbert's concepts that focus on physics formalization in the discipline of mathematics. The original version from Hilbert's statement is translated from German which states: mathematical consideration of physics axioms. It seeks to investigate the basis of geometry through treatment of the mathematically related physical sciences with the help of axioms. The good examples of this concept are captured in the theories of mechanics and probabilities. This problem is the only one that has been progressively engaging; at the moment most concepts in physics have been wholly formalized while the rest are in the process of being comprehended in a manner that is systematic (Zhang, 2017).

Hilbert's initial instance of probability theory is regarded as axiomatized from the'30s through the ideologies of the measure theory. For instance, in mechanics, the formalization status critically relies on the manner in which one comprehends the term currently. The particular classical mechanics' case referred to by Hilbert has been formalized fully through variation calculus and symplectic geometry. In conjunction with Einstein, Hilbert contributed to gravity formalization and further developed the concept of quantum mechanics as a refinement formalization of classical mechanics (Bendkowski et. al., 2017).

Although to date quantum mechanics' ontological status as physics theory is still unknown, its mechanics are captured by operator algebra and functional analysis theory. However, this circumstance varies drastically when one shifts to quantum field theory which has been understood since the ‘50s as a general and fundamental mechanics version in nature. From 1960 the AQFT (Haag-Kastler axioms) have been suggested as a formalization of the field theory and essentially for the local-QFT. Whereas these axioms were somehow successful in building a foundation for the structural outcome like the PCT theorem, the challenge lies in their continuous lack of corresponding models in dimensions beyond two (Berger & Hou, 2017).

The current developments in higher geometry and algebra indicate that axioms refinement is natural and necessary to the homotopy theory context. In the ‘90s the problem involving limiting of processes that resulted from the atomistic perspective of the motion laws of continua was tackled by several mathematicians. Generally, it was a declaration of the axiomatic methodology of expansion outside the prevailing mathematical disciplines; in physical sciences and beyond (Castellan et. al., 2015). This type of expansion needs semantics development of physics with a formal evaluation of the physical reality notion that should be performed. The two fundamental models aim at capturing most of the basic physics phenomena include the QFT which gives the standard model's mathematical framework (Bendkowski et. al., 2017). The other theory is that f general relativity that offers a description of gravity and space-time on a microscopic scale. However, the arguments are not in essence logically consistent; thus indicating the necessity of the anonymous quantum gravity theory.

A content review on this problem suggests that it emphasizes in mathematical formulation that involves quantum mechanics (Slemrod, 2013). They allow for the rigorous description of the mechanics and differ from the earlier aspect by the use of mathematical structures abstraction like the infinite dimensional operators and Hilbert spaces. Most of the structures are derived from a research field in pure mathematics known as functional analysis that was affected by the necessities of quantum mechanics (Castellan et al, 2015). It is clear from the study that natural morphisms like momentum and energy were regarded as eigenvalues rather than phase space function values; precisely Hilbert's space linear operators' of spectral values.

Hilbert's idea was an introduction of a category of axioms that could provide an explanation for an extension of physical phenomena class and progressively add forms to bring it closer to the reality (Li et al., 2017). In every step of axiomatization, it is important to give scientific evidence that all the previous results are still valid and consistency of the resultant axioms with an impact of Godel's theories but no specified way to derive the axioms.

Primarily deriving axioms is related to the particular relativity theory; it is done by taking the invariance of light speed and the principle of relativity which is formulated in mathematical terms. From the underlying geometry of the derived axioms, it is possible to determine special relativity. According to Hilbert, the primary focus was on statistical mechanics which evolved to quantum mechanics where thermodynamic limits and mean values were utilized to achieve results yet they lacked solid mathematical foundation (Zhao et.al., 2018). From past studies it is evident that Hilbert's sixth problem was entirely centered on physics that is mathematically rigorous; which implies that manipulations begin with some axioms and then everything progresses (Gorban, 2018). He explicitly stated that it would be necessary to axiomatize physical theories that are patently false since they might not be covariant or not quantum.

CHAPTER THREE
MATERIALS AND METHODS

3.1 Introduction

If the arrows in Cartesian product are reversed, one gets sum. It can be shown that in this manner, a Cartesian closed category can be converted to a ring. Kock (1981) showed that a derivative can be defined in a ring with infinitesimals. For the derivative to be useful in optimization theory, it should be defined in a ring (Cartesian closed category) with infinitesimals in a form that is similar to that of Sukhinin (1982). The idea of boundedness in Cartesian closed categories is necessary in order to come up with a criterion of obtaining extrema in Cartesian closed categories with infinitesimals. The next step will be to define an extremal point in a Cartesian closed category with infinitesimals and finally prove a result on the conditions for the existence of extrema. In this regard, the study intends to formulate and prove Theorem 2 in Cartesian closed category, then proceed to show that the theorem can be given in the form of the rule of Lagrange's multipliers

3.2 Important definitions and criteria used

The method in which the study intends to use to get the optimization results in Cartesian closed categories hinges on the following definitions:

3.2.1 Infinitesimals

Infinitesimals are the collection of objects $\{d \in D : d^2 \neq 0\}$. Let R be a field, D the collection of infinitesimals in R. A map $r : D \to D$ will be called a homomorphism of infinitesimals if $\forall d \in D, d^{-1}r \in D$.

3.2.2 Small functor

Let $x \in K \subset R, S \subset R$ where R be a ring with ideal. A functor $r : S \to R$ will be called D small relative to the cone K at the point x_0 if $\forall d \in K, \exists \lambda : x_0 - \lambda x \in S, r(x_0 - \lambda x) \subset D$. D is an ideal.

3.2.3 Differentiable functor

A functor $f : S \to Z$ is called D differentiable relative to the cone K at the point x if $f(x+d) - f(x) = Ad + d$ where A is the derivative of $f(x)$ and $d \in ideal$ of R.

3.2.4 Conditional extremum of a functor

Let $x \in S \subset R,$ where R be a ring with ideals and let $f : S \to Y$ be a functor satisfying $f(x_0) = 0.$ We say that x_0 is a point of conditional extremum of the functor $F : S \to R$ relative to the cone K under the condition $x \in K$ provided $F(x_0)\, Ø\, F(S)$ or $F(x_0) \in \partial F(S)$

3.2.5 Extremal Point

Let X be a ring with ideal, f be a morphism in X , $x \in X_0$ is an extremal point of the morphism if $f(x_0) \subset \partial X$

Using te definitions, it should be possible to get a result similar to Lagrange's method of multipliers in Cartesian closed categories by modifying Sukhinin's method.

CHAPTER FOUR
RESULTS AND DISCUSSION

4.1 Introduction

Sukhinin (1982) carried out a study on the Lagranges multipliers in vector spaces with bounded topology. Sukhinin (1982) used the idea of infinitesimals to obtain optimization results in vector spaces with bounded topology. It is this idea that this study genaralizes to Cartesian closed categories. In this study, a ring with ideals is taken to be a Cartesian closed category since it satisfies all the properties of a Cartesian closed category and as such it is a Cartesian closed category. To obtain the results similar to those obtained in classical analysis, Sukhinin (1982) used the idea of cones in linear topological spaces without norm. We follow Sukhinin (1982) to obtain the following important definitions;

4.2 Cone in a ring with ideal

4.2.1 Cones

Let R be a ring with ideal. $K \subset R, U$ a neighborhood of zero and B a bounded neighbourhood of zero in the ring R. Take a fixed $x_0 \in R$, then $K = \{d \in R : x_0 + t(d + U \cap B) \subset R\}$ and $K_+ = R \setminus K$ are cones.

4.2.2 Functor

Let $x \in K \subset R, S \subset R$ where R be a ring with ideal. A functor $r : S \to R$ will be called D small relative to the cone K at the point x_0 if $\forall d \in K, \exists \lambda : x_0 - \lambda x \in S, r(x_0 - \lambda x) \subset D$. D is an ideal.

4.2.3 D differentiable functor

A functor $f : S \to Z$ is called D differentiable relative to the cone K at the point x if $f(x+d) - f(x) = Ad + d$ where A is the derivative of $f(x)$ and $d \in ideal$ of R.

4.3 Synthetic Differential Geometry in Cartesian closed categories

This section aims at creating differential structures that are in line with constructive mathematics. The main purpose of this approach is to get results that are related to optimization that avoid restrictions imposed by Godel's Incompleteness Theorem. By Church's thesis, this approach can be modelled by computer programs (Dershowitz & Gurevich, 2008). Constructive mathematics takes a proposition to be true if it can be constructed. This is why we define a derivative in an algebraic setting in this section.

In this study, we discuss the synthetic differential geometry axioms in Cartesian closed category. The objects in Cartesian closed category according to synthetic differential geometry axioms have a structure that is differentiable, each arrow has a derivative, and the basic rules of calculus are calculations of the rings with ideals. In this study, it is assumed that there is a ring with ideal that satisfies the synthetic differential geometry axioms. The objects are categories and the arrows are functors, and to global elements $p:1 \to R$ as points of the ring R. The axioms posit a ring R, with addition and multiplication making a kind of a number line. But R is unlike the standard reals in that the set of ideals $D \to R$ of $x \in R$ with $x^2 = 0$ is not just $\{0\}$. D is called the set of ideals with square zero and every functor from the set of ideals to the ring R has a well-defined slope.

4.3.1 Derivative of a ring with ideal

If f is a ring homomorphism, and

$f(x+d) - f(x) = Ad + d$

Then A can be called a derivative. $x, d \in R$ Where $d \in ideal$ of R.

4.3.2 Proposition

If g is a ring homomorphism then it maps ideal to ideal.

Proof

Since $ring \times ideal = ideal$ and g is a ring homomorphism from $X \to Y$ then

$g(ring \times ideal) = g(ring)g(ideal) = g(ideal)$

Therefore, $g(ideal) = ideal$

Hence A is a derivative of f at x.

For example, compare this with the usual Frechet derivative $f(x+h) - f(x) = Ah + r(h)$ where Ah is an infinitesimal. In our case, ideal takes the place of infinitesimal.

4.3.3 Consequence of the definition

Suppose that f is a constant, that is, for some $c \in R$ we have $f(x) = c \forall x$ then,

$f(x+d) = c = f(x)$.

Therefore, $f(x+d) - f(x) = Ad + d$ implies that $Ad + d = 0$ or $(A + id)d = 0$,that is , $d = 0$

But $Ad = 0$ since $A(0) = A(d-d) = Ad - Ad = 0$

4.4 Derivative properties

4.4.1 Theorem 1

If A is a derivative then $A(f.g) = f.A(g) + A(f).g$

Proof

Following Mac Larty (1992)we show that $A(f.g) = f.A(g) + A(f).g$

Indeed $A(f.g) = f(x+d)g(x+d) - f(x)g(x)$

$$= f(x+d)g(x+d) - f(x+d)g(x) + f(x+d)g(x) - f(x)g(x)$$

$$= f(x+d)[g(x+d) - g(x)] + [f(x+d) - f(x)]g(x)$$

$$= f(x+d)[Ad+d] + [Ad+d]g(x)$$

$$= f(x)Ad + f(x)d + f(d)Ad + f(d)d + Adg(x) + dg(x)$$

$$= fA_g d + d + 0 + 0 + gA_f d + d$$

$$= d(fA_g + gA_f) + d$$

$$= fA_g + gA_f$$

Hence the proof.

4.4.2 Theorem 2

If A is a derivative and suppose f and g are ring homomorphisms which are differentiable then

$A(f+g) = Af + Ag$

Proof

Indeed,

$$A(f+g) = [f(x+d) + g(x+d)] - [f(x) + g(x)]$$

$$= [f(x+d) - f(x)] - [g(x+d) - g(x)]$$

$$= [A_f d + d] + [A_g d + d]$$

$$= Af + Ag$$

Hence the proof.

4.4.3 Theorem 3

If A is a derivative and suppose that f is a homomorphism and c is a constant then,

$A(cf) = cAf$

Proof

$$A(cf) = c[f(x+d) - f(x)]$$

$$= c[A_f d + d]$$

$$= cAf$$

Hence the proof.

4.4.4 Theorem 4

If A is a derivative and f, g are homomorphisms in a ring R, then

$$A\left(\frac{f}{g}\right)=\frac{gAf-fAg}{g^2}$$

Proof

$$A\left(\frac{f}{g}\right)=\frac{f(x+d)}{g(x+d)}-\frac{f(x)}{g(x)}$$

$$=\frac{g(x)f(x+d)-g(x+d)f(x)}{g(x+d)g(x)}$$

$$=\frac{g(x)f(x+d)-g(x)f(x)+g(x)f(x)-g(x+d)f(x)}{g(x+d)g(x)}$$

$$=\frac{g(x)[f(x+d)-f(x)]-f(x)[g(x+d)-g(x)]}{g(x+d)g(x)}$$

$$=\frac{g(x)[A_f d+d]-f(x)[A_g d+d]}{g(x+d)g(x)}$$

$$=\frac{(gA_f)d-(fA_g)d}{g^2}$$

$$=\frac{gAf-fAg}{g^2}$$

Hence the proof.

4.5 Proposition 1

Let f be a differentiable functor in a cone K with the vertex at x_0, then $f(K)$ is a cone with vertex at $f(x_0)$.

Proof

$$f\{x_0+t(d+U\cap B)\}-f(x_0)=A\{t(d+U\cap B)\}+(d+U\cap B)$$

But since $t(d+U\cap B)\in K$ and A is linear then $A\{t(d+U\cap B)\}$ is an open cone. Hence the R.H.S is a cone.

$\therefore$ L.H.S is a cone with vertex at $f(x_0)$

Hence the collection of cones with differentiable maps as morphisms form a Cartesian closed category.

4.6 Proposition 2

Let $f: K\to K$ be differentiable, if $x\in K$ and $Ax=0$ then $f(x)\in\partial K=f(x_0)$.

Proof

$f\{x_0 + t(d + U \cap B)\} - f(x_0) = A\{t(d + U \cap B)\} + (d + U \cap B)$, then

$f\{x_0 + t(d + U \cap B)\} - f(x_0) = (d + U \cap B)$

$\Rightarrow f\{x_0 + t(d + U \cap B)\} \approx f(x_0)$

By proposition 1, $f(x_0)$ is a vertex and hence $f : x \to f(x_0) = \partial f(K)$

4.7 Optimization in Cartesian Closed Categories

Adjoint in categories has been suggested by Lawvere (1981) as an optimization in categories. Although it is thought that the interpretation of this optimization should not be taken to be similar to the one in analysis, this thesis shows that it is a generalization of the usual optimization.

4.7.1 Adjoint

Let X, Y be categories and $F : X \to Y, G : Y \to X$. If $F \circ G = 1_Y, G \circ F = 1_X$ then F and G are adjoints of one another.

4.7.2 Sukhinin on Extremum

Let X be a topological category and ∂X be a category of boundaries. If $F : X \to \partial X, G : \partial X \to X$ and $F \circ G = 1_{\partial X}, G \circ F = 1_X$ then F and G are said to be the optimizations of the categories.

From the definition, it can be noted that retract is an optimization and derivatives and antiderivatives are optimizations of each other.

4.7.3 Conditional extremum of a functor

Let $x \in S \subset R$, where R be a ring with ideals and let $f : S \to Y$ be a functor satisfying $f(x_0) = 0.$ We say that x_0 is a point of conditional extremum of the functor $F : S \to R$ relative to the cone K under the condition $x \in K$ provided $F(x_0)$ Ø $F(S)$ or $F(x_0) \in \partial F(S)$

4.7.4 Theorem 5

Let X and Y be rings with ideals, $S \in S', K_1$ a cone with vertex at zero. $f : S \to Y$ satisfies the condition $[x_0 + dh + x] \wedge [f(x_0 + dh + x) = f(x_0)]$ and $f(x_0) = 0$. Further, let Z be a ring with ideal, $F : S \to Z$ be $S_D^{'}$ differentiable relative to K_1 at x_0 . If x_0 is a point of conditional minimum of F relative to the cone K_1 under the condition $f(x_0) = 0$, then $F'(x_0)h \notin t(h + U \cap B)$ for $h \in K_1$.

Proof

Assume $x_0 = 0, F(0) = 0.$ Let $h \in K_1$ and $V \subset D$ in $Z, r(x) = F(x) - F(0).(x)$ and $V' \subset D: V' + V' \subset V$. Then $\exists U_1$ and $\partial \in D: F'(0)(U_1) \subset V' \forall B \forall d \in D$.

Let $x' \in [d(h + U_1 \cap B)] \cap S'_D, f(x') = 0$ then $[r(x') \in dV'] \wedge [F(x') \in t(h + U \cap B)]$.

Further, $\exists x \in d(U_1 \cap B)$ for which $(dh + x) \in C$, where C is a positive cone in Z. Further,

$F'(0)(dh + x) + r(dh + x) \subseteq dF'(0)h + F'(0)x + r(dh + x)$ then

$F'(0)h + F'(0)(d^{-1}x) + d^{-1}r(dh + x) \in [F'(0)(h) + V' + V'] \cap C[F'(0)(h) + V] \cap C$, that is,

$[F'(0)h + V] \cap C \neq \varnothing,$ since C is closed and V is arbitrary, $F'(0)(h) \in C$, that is

$F'(0)h \notin t(h + U \cap B).$

Sometimes $F'(x_0)h \notin t(h + U \cap B)$ can be given in the form of the rule of Lagrange multipliers. Let Z be a ring with ideal, g be a ring homomorphism from X into Z, A be a functor from X into Y, K be cone in X and suppose that;

$\forall h \in KerA \cap K : g(h) \in d(h + U \cap B)$, $AX = Y$ and $A = X$ then there exists a functor $\lambda \in \aleph(Y, Z)$ such that $(g + \lambda \circ A)X_0 = 0$ for some $X_0 \in X.$ Where $\aleph(Y, Z)$ is a class of all functors from Y into Z.

4.8 Applications

4.8.1 Building Spreadsheet Application

The excel spreadsheet application lacks specific features that aid in generating modular designs. If it had these specifications like the majority of programming languages, it would be time-saving as well as minimize mistakes, facilitate organization style, and ease code control. There are two critical approaches utilized in software development, where one involves ensuring the design is too simple to have deficiencies while the other entails the use of a complicated procedure to eliminate inadequacies. Since the former methodology is a relatively more difficult, the only strategy to construct complex applications that have no limitations is through a combination of simple modules by proper interfaces (Sadaphule & Shaikh, 2016). This concept would make most of the emerging issues local therefore creating the hope of optimizing or fixing a specific part with no breaks in the entire program. Modularity increases the alternatives for system modification and is the primary outcome of the options theory, which claims that it is better to have a group of only options rather than a single choice within a given portfolio. Generally, it is possible to model the modularity value in a manner that is decision-theoretic as seen in the Blak-Scoles equation.

Intuitively, applications whose constituents can get upgraded selectively are relatively more important than a program that is a monolith. For this selective approach to be possible it is necessary to ensure that the program modules have a modular design; hence the need for a semantic property referred to an abstraction. Increased modularity would also give an assurance of useful bug identification to avert affecting an arbitrarily significant system proportion through a process known as compartmentalization. The well-defined interfaces are used to enforce the compartmental boundaries efficiently. These separation mechanisms would enable reliable prevention of the missed bug for executing errors in a more significant proportion of the whole system.

Spreadsheets lack an essential feature utilized in controlling system complexity regarding defining the abstractions that are reusable which is very crucial for end-user coders. The excel sheet applications are designed through mathematical treatment of the spreadsheets using the Excelsior programming language to create modularity. The language develops by the modularity principles by implementing the mathematical functions and arithmetic operators; it is an imitation of python as a useful technique that is concise and dynamic typing (Sadaphule & Shaikh, 2016). Building spreadsheet applications is made possible by applying the design principles and its user-interaction as well as facilitates trial an error placement of results produced in the process of structure discovery.

4.8.2 Neuroscience

The art of neuroscience is facing a challenge of identifying the suitable language to provide a proper description of the brain's activity in a manner that results in theory evaluation, deduction, and also a calculation. There is need to bridge the gap between neuroscience and literature as thoughts, concepts, percepts, emotions, and ideas; hence it is necessary to develop a mathematics concept that attempt to resolve these problems from the neuroscientists and mathematicians' perspectives (Ashby & Valentin, 2017).

The contribution of mathematics in this field is in the manner in which math works on process description like concept refinement and abstraction. The category theory is among the most influential mathematics developments that have made significant advancements in the manner in which it describes mathematics processes. It is also helpful in the unification of various topics, new logics development and revelation of the general procedures that have translated to broader uses as well as implications. Abstraction enables the use of analogies through correlation of encoding as well as relations between relations.

It may be presumed that abstraction power, in mapping ought to get deeply encoded in the history of evolution as a survival technique, since maps result in an environment model that is small and can be manipulated. In the mathematical manipulation of representations entails typically rewriting, for instance, the application of the commutative law which is complicated formulae replacement (Ashby & Valentin, 2017). This study of the manner in which mathematics operates is crucial in model development to support neurological functions involving environment maps.

The principles of mathematics could potentially offer a case study that is comprehensible regarding complicated interacting structures evolution and can result in useful analogies in development and analysis of brain activity models. The concept of category theory is relevant to biological studies; specifically, the colimit notion in any category in describing structures that consist of interrelated components (Sadaphule & Shaikh, 2016).

Therefore, a category that evolves with time can then enable structures undergoing evolution; and the compositions are provided as the Ct at the time, t. The aspect of colimits provides an overall setting whereby it describes the amalgamation process of complex models. The category notion developed from helpful function notation which shifts from the vague definition of $y = f(x)$ to a more clear function $f : X \to Y$, which makes the equation more of a process than a mere function. The category is composed of a class of objects, arrows, and composition and in addition to it also has position structure defined by its object's class. The applicable rules are associativity and presence of identities; in short, the colimit notion in every category generalizes the union formation perception of overlapping objects with a defined intersection. This approach is applied in the human brain whereby information from various sensory organs is reintegrated in mind to be geometrically sensible.

4.8.3 Cognitive neural network architectures

The category theory is applicable in mathematical modeling of the cognitive neural networks semantics with the help of functors, colimits, as well as natural transformations to support structural mapping of the theory. In this case, the functors map the colimits concept onto a neural components category; natural transformation approach among the functors amalgamates the idea of single-sensor symbolism in a joined, multi-mode cognitive neural network design. These mathematical schemes address the underlying semantics in an organization's neural systems.

When a neural network coupled to the sensor(s) gets stimulated by actions that require simple representation of the component concept (Ashby & Valentin, 2017). The anode in the network receives activated on reception of adequate input through a specific subset of the excitatory connections of the information to prevent data from inhibitory connections. Once it has been activated, the node sends a signal, output, to the others or an endpoint device as a signal function. The activation process is based on the input connection weights both the inhibitory and excitatory and is dependent on the node's activation potential as well as the threshold.

The category theory has primitive quantities that include morphisms and objects; whereby morphism consists of an object that is the domain, and the other is referred to as the co-domain. In a given category, each of the arrows $g: c \to a$ and $f: a \to b$ poses an arrow of composition $g \circ f: c \to b$ with a domain a and codomain c; thus satisfying the associative law. The fundamental concept in the analysis of the category theory is the commutative diagram, which provides a categorical equivalent of the equations used in the system (Ashby & Valentin, 2017). The layouts are of great significance to the argument since they play the role of specifying the structures' constraints. One of the major applications is in the definition of colimits used structural mapping of the category concepts in neural components (Sadaphule & Shaikh, 2016). The category concepts have morphisms and objects as concepts that define specifically the way the real part represents a subconcept while the imaginary for the logical part.

In addition to mapping operations and sorts, the category theory structure has the property of axiom refection. These morphisms' compositions in a concept are a combined relationship of sub-concepts; is a commutative representation of colimits in assembling formalization. A neural system is composed of several helpful commutative diagrams with the reciprocal network (Ashby & Valentin, 2017). The feature implies that each connection has similar polarity; that is the inhibitory and the excitatory for any given node in the neural system. The structure allows for functional mapping of the neural concept structure

4.8.4 Program Optimizations

In the world of computer science, software optimization refers to the modification process of the program to improve the efficiency in performance of a particular aspect of the software as well as utilizing the use of computer resources. Generally, computer software could get optimized to ensure it executes rapidly or minimize storage and power consumption as well as other resources. This process involves identifying the aspects of a program that could be discarded and those that are important.

The category theory applies to the generalization of both the original and the changed codes into widely used optimization rules (Elliott, 2017). The category formalization is a necessary abstraction technique used in learning program optimization for correcting typing bugs using a relational database to help programmers in code compiling. Most program optimization techniques entail direct style transformation to a similar program through continuation passing approach. However, the most convenient way to achieve such computation is through a set of changes that preserve meaning being added to the available optimization rules for instance use of GHC in Haskell programming. Optimization is a desirable functionality in software development but is usually antagonistic to the goals of portability, maintainability, and stability (Elliott, 2017).

In these functional programming languages, such as implementation illustrated in the various interpretations such as the automatic differentiation, hardware circuits, interval analysis as well as the incremental computation. From a category-theoretic perspective, it is evident that the optimization capacity offers a reasonable option for the domain-particular languages that are deeply embedded. The initial phase of compiling to the category theory involves performing syntactic transformation which changes language from simple calculus to the form of Cartesian closed categories without a change in meaning (Elliott, 2017). The point of this transformation is to facilitate easy conversion from the parent category to the other classes, which are the interpretation options of the directly typed calculus and therefore the Haskell programs. The aspect of changing vocabulary without varying the intended meaning through the use of homomorphism equations is fundamental to compiling optional categories; hence giving a non-standard and sound interpretation to the functional codes.

CHAPTER FIVE
CONCLUSION AND RECOMMENDATION

5.1 Conclusion

The rule of Lagrange's method of multipliers has been appied to topological spaces without norm by Sukhinin (1982). Otieno et al (2013) generalized the theorem to ordered topological modules. In this work, the theorem has been applied to Cartesian closed categories for the first time.

Within the same space it may be possible to study optimization using the Lagrange's method of multipliers. Once the cones were defined, the derivative from which the extrema was obtained was realized. It was hence possible to define a differential structure in Cartesian closed category. To induce a global differential structure on the space induced by homeomorphisms, their compositions on chart intersections in the atlas must be differentiable functions on the corresponding space.

Further, the ability to induce a differential structure on a Cartesian closed category allows for an extension of the definition of differentiability to spaces without the global coordinate systems. A differential structure would allow one to define the globally differential tangent space, differentiable functions, differentiable tensor and even vector fields.

Optimization is very important in physics and optimization in cartesian closed categories by Lagrange's method of multipliers is vital since it ensures that optimization problems are easily solved. The optimization results obtained in this study were applied to nonlinear differential equations using algebraic geometry techniques.

5.2 Recommendation

Differential equations are central to all areas of physics and this study has made an attempt to apply optimization results to non-linear partial differential equations. Further research is recommended to study on how category theory can help solve differential equations by mapping diagrams of equations to other categories similar to how problems of topology are often solved by mapping topological spaces to algebraic ones in algebraic topology. Further research should also be done on the use of Cartesian closed categories in the formulation of programs that will aid in proving mathematical theorems.

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