INTEGRATING FUZZY AND BIPOLAR FUZZY METRICS INTO DIGITAL TOPOLOGY FOR ENHANCED IMAGE ANALYSIS

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A Thesis Submitted to the Graduate School in Partial Fulfilment of the Requirements for the Doctor of Philosophy in Pure Mathematics Degree of Egerton University

EGERTON UNIVERSITY

OCTOBER, 2025

DECLARATION AND RECOMMENDATION

Declaration

This thesis is my original work and has not been presented in this University or any other for the award of a degree.

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Recommendation

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DEDICATION

To my incredible wife, for your unwavering support, patience, and for pretending to find fuzzy theory as exciting as I do. And to my awesome children, for reminding me that no matter how complex my research gets, there is nothing greater than the challenge of raising you up in godly ways!

ACKNOWLEDGMENTS

First and foremost, I owe everything to God – the ultimate supervisor of all of life. Without His wisdom, strength, and those unexpected bursts of inspiration at odd hours, this work would have been a distant dream. Truly, all glory belongs to Him.

To my supervisors, Dr. Gichuki M. N. and Prof. Sogomo K. C., thank you for your patience, constructive criticism, and for somehow believing in this research even when I had moments of doubt. Your guidance and relentless push for excellence kept me on my toes (and often in front of my laptop for far too long). I sincerely appreciate your time, mentorship, and unwavering support.

A special thank you to my wonderful wife, who not only tolerated my late-night ramblings about fuzzy metric spaces but also managed to keep me sane through it all. Your love, encouragement, and occasional reminders to eat and sleep have been invaluable. You truly are the real MVP of this journey.

To my amazing children, Ana, Jabari (a placeholder for Pebo) and Kala, you probably wondered why Dad was always glued to his computer instead of playing more games with you. Well, here's the answer! This work is dedicated to you as proof that persistence and hard work (plus a little bit of coffee) can achieve great things. I promise to make up for all the missed playtime.

To my Dad and Mum – we did it. From the very beginning, you taught me the value of education, discipline, and faith. Your sacrifices, prayers, and unwavering belief in me laid the foundation for everything I have achieved. Every late night and every milestone in this journey was fueled by the values you instilled in me. This achievement is as much yours as it is mine. Thank you for your endless love, encouragement, and for always reminding me that no dream is too big when guided by purpose and perseverance.

To my friends and colleagues, thank you for pretending to understand when I excitedly explained my research – your polite nods and enthusiastic "oh, that's interesting" responses were much appreciated! Your support, encouragement, and occasional distractions helped me stay sane.

ABSTRACT

Digital topology provides a mathematical foundation for analysing discrete image structures. It is considered to be a very critical part of processing image data because it provides us with information about the properties of images. The approaches to digital topology have been studied to great detail by several researchers and each approach has been seen to have advantages and limitations. In general, the approaches that exist, which are the Graph theoretic and axiomatic methods, are both found to struggle when it comes to handling the imprecision that is present in images and other digital structures. This is because such methods do not account for dual uncertainty. Consequently, this limits their effectiveness in real-world applications. While fuzzy and bipolar fuzzy set theory provide flexible tools for addressing these limitations, their integration into digital topology remains underdeveloped. This study addresses this gap by developing a mathematical framework that extends traditional digital topology into fuzzy and bipolar fuzzy logic for image analysis. The research formalises fuzzy and bipolar fuzzy metric spaces on the digital plane, and attempts to integrate these bipolar fuzzy metrics into digital topology. Topological properties, including connectedness, adjacency, and surroundness, are also extended into the fuzzy and bipolar fuzzy domain. This has been done in order to ensure mathematical consistency in representing digital images. Additionally, the study proposes new similarity, distance, and entropy measures to refine the important aspects of image analysis such as image segmentation. Through computational implementation, the findings contribute to a more realistic approach to image segmentation that is also mathematically sound. This research greatly improves the applicability of digital topology in image processing technique since it establishes a framework that handles uncertainty in digital images. The study's outcomes will pave the way for deeper analysis of digital images and hence strengthen the intersection of topology, fuzzy mathematics, and artificial intelligence.

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LIST OF ABBREVIATIONS

3-D	3-Dimensional Space
AI	Artificial intelligence
BFS	Bipolar Fuzzy Set
BFTS	Bipolar Fuzzy Topological Space
BFG	Bipolar Fuzzy Graph
IFM	Induced Fuzzy Metric
IFS	Intuitionistic Fuzzy Set
BRE	Bipolar Fuzzy Rényi Entropy
BJS	Bipolar Fuzzy Jaccard Similarity
BFE	Bipolar Fuzzy Entropy
FMS	Fuzzy Metric Space
$\mathbb{Z}$	Set of integers
ML	Machine Learning

CHAPTER ONE

INTRODUCTION

1.1 Background Information

Researchers are still finding the most rigorous framework that can define well-structured 3D digital spaces. This search has primarily followed two major approaches referred to as "the Graph-theoretic approach and the Axiomatic approach" (Edward & Lang, 2000). On one hand, the Graph-theoretic approach is mostly used in image and picture analysis because it is highly applicable and has direct applications in image processing (Chen & Zhang, 2004). On the other hand, we have the axiomatic approach which is considered to be more suitable for theoretical discussions. Even though the axiomatic approach is the most preferred approach, it remains underutilized in practical applications since it does not provide easy ways for doing computations (Eckhardt & Latecki, 2008).

In the axiomatic approach, some subsets of digital structures are said to be open sets and must satisfy topological axioms to ensure that the digital structure has properties correspoding to classical topology (Khalimsky *et al.*, 1990). The challenge, however, is in defining axioms that ensure both mathematical rigor and applications. Kovalevsky (1989) highlighted the limitation of existing axiomatic approaches when it comes to providing a sufficient mathematical framework that can also be used in application problems. As a result, several studies, including those by Kovalevsky (2006) and Khalimsky *et al.* (1990), have sought to develop alternative axiomatic models that present the fundamental properties of digital spaces in a much simpler way.

Digital topology is a branch of topology that studies topological properties in discrete digital spaces. These properties include pixel and voxel structures which are defined using appropriate adjacency relations and connectedness properties. Digital topology provides a rigorous framework for analysing continuity, connectivity, and boundaries in discrete spaces (Khalimsky *et al.*, 1990; Rosenfeld, 1979).

Definition 1.1.1 (Digital Topological Space)

A digital topological space is a pair (X, N_k) where $X \subseteq \mathbb{Z}^n$ is finite or countable discrete set of points and $N_k : X \to 2^X$ is a function defining a $k-$adjacency relation to each $x \in X$ (Kovalevsky, 2006).

1.1.1 Challenges in Digital Topology and the Role of Ersatz Topologies

Since digital images are finite and discrete, we apply locally finite topologies to them (Kovalevsky, 2006). However, elements of different dimensions require different neighborhood structures within the space and this can cause problems when performing computations. To address this challenge, researchers such as Khalimsky *et al*. (1990) have proposed graph-based approaches to digital topology. This alternative makes it easy for researchers to define adjacency relations that can be used to study connectivity in a way that is comprehensive.

Definition 1.1.2 (Digital Adjacency Relation)

To study nD digital images, we say that two distinct points $p, q \in \mathbb{Z}^n$ are $k-\left(\text{or } k(t,n)-\right)$adjacent if for $t \in \mathbb{N}$ $s.t$ $1 \le t \le n$ at most t of their coordinates differ by ± 1 and all the others coincide (Han, 2005)

We can obtain the $k-$adjacencies of $\mathbb{Z}^n$ as follows;

$$k = \sum_{i=n-t}^{n-1} 2^{n-i} c_i^{\,n}, \text{ where } c_i^{\,n} = \frac{n!}{(n-i)!\,i!} \tag{1.1}$$

1.1.2 Fuzzy Set Theory and Digital Topology

The introduction of fuzzy set theory was done by Zadeh (1965). The purpose of developing this theory was to provide a formal mathematical framework for handling ambiguous data. Imprecision, along with vagueness, is a common feature in real-world settings or phenomena and this requires a different approach to study as opposed to the classical set-theoretic methods. By extending classical set theory, fuzzy mathematics becomes well-suited for uncertain or noisy digital data.

Definition 1.1.3 (Fuzzy Set)

Let $A \in I^X$. The subset of X in which A takes non-zero values is called the support of A. $\forall x \in X$. Additionally, $A(x)$ is called the grade membership of x in A. X is called the carrier of the fuzzy set A.

If A takes only the value 0 and 1, then A is called the crisp set in X. $A \in I^X$ is said to be contained in $B \in I^X$, $A \leq B$, *iff* $\forall x \in X, A(x) \leq B(x)$ (Palaniappan, 2002)

Definition 1.1.4 (Bipolar Fuzzy Set)

Let X be a nonempty set. A pair $\mu = (\mu^+, \mu^-)$ is called a bipolar-valued fuzzy set (or bipolar fuzzy set) in X if $\mu^+ : X \to [0,1]$ and $\mu^- : X \to [-1,0]$ are mappings (Kim et al., 2018)

1.1.3 Fuzzy Metric Spaces and their Extension to Digital Topology

Fuzzy metric spaces have been further developed by various researchers, including Ege and Karaca (2015) and Han (2016). In their work, they establish a formal metric structure within fuzzy set theory. This allows for the measurement of gradual membership of elements in digital spaces, and ensures that digital structures can be analysed using continuous distance functions instead of only discrete ones.

Definition 1.1.5 (Fuzzy Metric Space)

The triple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t – norm and M is a fuzzy set on $X^2 \times [0, \infty)$ such that $\forall x, y, z \in X$ and $s, t > 0$, the following conditions are satisfied;

i. $M(x, y, 0) = 0$

ii. $M(x, y, t) = 1$ *iff* $x = y$ (1.2)

iii. $M(x, y, t) = M(y, x, t)$

iv. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

v. $M(x, y, \cdot) : (0, \infty) \to [0,1]$ is continuous

$M(x, y, t)$ is interpreted as the degree of nearness of x and y with respect to t

1.2 Statement of the Problem

The field of fuzzy metric spaces plays a crucial role in various domains by providing mathematical tools to quantify similarity and dissimilarity between objects. This process is done using distance functions or metrics. However, existing fuzzy similarity and distance measures primarily rely on single-valued membership functions. Therefore, this limits their capacity to represent dual uncertainty – that is, both the presence of a feature and its implicit counter-property within a dataset. Classical fuzzy metrics such as Euclidean distance and Jaccard do not account for the bipolar nature of uncertainty. This theoretical gap in fuzzy set theory restricts its effectiveness in fields that require relative similarity assessments, such as medical image segmentation. To address this limitation, this research develops bipolar fuzzy similarity, distance, and entropy measures that extend traditional fuzzy models by incorporating bipolarity. The study establishes a bipolar fuzzy metric space, defines new similarity and distance measures, and proposes a bipolar fuzzy entropy function for quantifying uncertainty in image analysis. These theoretical advancements are validated by applying them into bipolar fuzzy image segmentation techniques, demonstrating improved accuracy in grayscale image analysis compared to classical fuzzy segmentation models. The research provides a robust mathematical foundation for bipolar fuzzy digital topology, ensuring that these concepts can be effectively applied in digital image processing and other real-world computational problems.

1.3 General Objective

To establish a framework for integrating fuzzy and bipolar fuzzy metrics into the digital plane and demonstrate the application of these results in image analysis.

1.3.2 Specific Objectives

i. To formulate the theory of fuzzy and bipolar fuzzy metrics for digital topology and show their mathematical completeness

ii. To adapt key topological properties, such as connectedness, adjacency, and surroundness, into fuzzy and bipolar fuzzy digital spaces.

iii. To construct novel similarity, distance, and entropy measures for use in image analysis and pattern recognition.

iv. To conduct experimental validation of the fuzzy and bipolar fuzzy theory in digital image processing.

1.4 Justification

Digital topology provides a rigorous mathematical foundation for image analysis by providing ways by which problems in digital structures can be solved. It is considered to be a useful tool in fields that require detailed analysis of digital images. However, the need to expand the foundation on which the theory is developed, has been recommended by many researchers in this area. This need arises from the fact that the existing approaches are rigid and do not have a wider scope of application. Therefore, it is essential to define and formalise new approaches that can accurately capture the realities of subjectivity that exist in real-world contexts. These principles are best defined within fuzzy and bipolar fuzzy frameworks. By incorporating bipolar fuzzy set theory into digital topology, this research establishes a more flexible framework that better captures the dual nature of information. This study specifically addresses the gap in existing literature by formulating and characterising distance and similarity measures used on bipolar fuzzy sets. The significance of this study extends beyond theoretical results. It offers methodologies that may be used in several computational and analytical processes and that are shown to enhance the accuracy of systems that deal with uncertain information.

1.6 Limitations

i. Although this research develops a rigorous mathematical foundation for bipolar fuzzy metric spaces practical implementation on large or complex datasets may present challenges, particularly when it comes to computational efficiency and scalability.
ii. The bipolar fuzzy similarity and segmentation algorithms proposed in this study may require greater computational resources than classical fuzzy approaches. This could limit their suitability for complex image processing tasks where efficiency is critical.
iii. Some of the introduced measures, including Bipolar Fuzzy Entropy and Bipolar Rényi Entropy, rely on tunable parameters. Determining appropriate parameter values for specific applications requires testing, which may introduce a degree of subjectivity.

iv. The framework assumes that positive and negative membership functions sufficiently capture dual uncertainty within a dataset. However, defining accurate membership functions can be difficult especially where data does not show clear opposing characteristics.
v. While the study presents promising results in areas such as image processing and medical imaging, further validation across other application domains is needed. We propose this as a way of assessing the broader application of the proposed model.

These limitations have been extensively in Chapter Four of this thesis

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

In many ways, mathematics is recognised as a cornerstone of science, especially in technological innovation (Shen et al., 2012). Its applications cut across numerous disciplines and this makes it a vital tool in solving complex real-world problems that appear in different settings. Over the years, notes Zimmermann (2001), mathematical theories have evolved in order to adapt to the many needs in different fields. Fields such as physics, engineering, economics, and computer science have benefited greatly following advances in mathematical theories. In particular, one such area of advancement is digital topology. This is a field of science that provides a mathematical framework for analysing discrete structures, particularly in image processing and computational modeling. It is this context that forms the foundation of this research (Zimmermann, 2001).

Digital topology is an extension of classical topology into the digital plane (Han, 2016). In this plane, digital images and spatial data are defined and studied. This field of topology is useful because traditional continuous methods are considered to be not inadequate in studying images. According to Han (2016) this is because classical mathematical models do not accommodate the nature of image data which is considered to be very variable. To address this limitation, fuzzy set theory has been used to study the digital plane because it provides a means of extending conventional mathematical ideas used in digital topology into areas that are considered to be vague (Klir & Yuan, 1995).

This chapter provides a critical evaluation of existing literature in fuzzy mathematics as well as in digital topology. We will highlight the gaps and challenges in current research and emphasise the need for an approach that can combine both fuzzy mathematics and digital topology. By establishing this foundation, the chapter aims to set the stage for the development of digital fuzzy and bipolar fuzzy metrics and these concepts will be very essential when it comes to enhanced image analysis.

Further, by combining digital topology and fuzzy mathematics, researchers have found ways that can be used to extend set-theoretic approaches to accommodate uncertainty and imprecision. This chapter will also discuss the origins of the concepts of fuzzy set theory and fuzzy metric spaces.

Afterwards the discussion will highlight the axiomatic and graph-based approaches to digital topology more closely and show how different properties of digital images are defined in each of the approaches. The discussion will also highlight some of the important theoretical advancements in both fuzzy theory and digital topology and also identify gaps that still exist in literature. The relevance of digital fuzzy metric spaces in modern computational systems will also be highlighted.

By providing a structured review of these topics, this chapter lays the groundwork for further exploration of digital fuzzy and bipolar fuzzy topologies and their practical significance.

2.2 Fuzzy Set Theory

The theory of fuzzy sets has been described by researcher to be, “a theory in which everything is a matter of degree” (Zimmerman, 2001). Classical approaches to modeling real life phenomenon are rigid (Zimmerman, 2001). For instance, in binary logic a statement is either true or false, and nothing in between. In sets, an element either belongs to a set or does not. In optimisation, a solution is either possible or not. This nature of mathematics carries the assumption that a model can fully describe a physical phenomenon without leading to ambiguities (Zimmerman, 2001). It has been argued that this assumption is not sufficient.

Concepts that are defined in the traditional sense are considered to be certain because they are developed using binary logic. This certainty is only going to be true as long as parameters under discussion are known to be fixed. Consequently, this means that there cannot exist any doubt regarding the values of elements in the sets under consideration. In the same way, occurrence of such phenomena will also be known with certainty (Zimmermann, 2001). This does not imply that a mathematical theory that treats elements to be precise is irrelevant. Instead, it just means that utility of such theory will only be evident in scenarios where the constructed model is not intended to represent reality. In such cases, parameters can be selected arbitrarily (Zimmermann, 2001).

The limitations of set theory are therefore more visible when modelling real-world phenomena that is vague or ambiguous. As noted by Popper (1959) and Shen et al. (2012), “rigid, dichotomous logic often fails to capture the complexity of human thought, perception, and decision-making, which rarely conform to absolute categories” (p. 124). In summary classical mathematical models will therefore always fail to show both aspects of a phenomenon. For that reason, they are considered to be rigid even when a flexible structure that reflects the uncertainty of real-world phenomena would be more appropriate.

To bridge this gap, fuzzy set theory offers a more a framework which allows for partial membership of elements in a set. It therefore enables mathematical models to handle partial truths and uncertainty more effectively. Unlike crisp logic, fuzzy logic preserves stability even under minor variations in assumptions. By incorporating fuzzy logic, mathematical models gain the ability to represent reality more accurately.

2.2.1 Fuzzy Set and Fuzzy Point

To define a fuzzy set, it is important to assign each possible individual in the universe of discourse with a value that represents grade of membership (Zimmermann, 2001). This grade corresponds to the degree to which that individual is similar or compatible with the concept represented by the fuzzy set.

The definition of a fuzzy set is formalised in Palaniappan (2002) by taking the approach provided below. If $I = [0,1]$ and $X \neq \emptyset$ then a fuzzy set is a function with domain X and values in I, that is, an element of I^X.

Let $A \in I^X$. $\forall x \in X$, $A(x)$ is called the grade membership of x in A. If A takes only the value 0 and 1, then A corresponds to the crisp set definition of a subset of X. $A \in I^X$ is said to be contained in $B \in I^X$, $A \leq B$, *iff* $\forall x \in X, A(x) \leq B(x)$.

2.2.2 Notations

Two distinct notations are commonly used in literature to denote membership functions (Zimmermann, 2001). Firstly, the membership function of a fuzzy set A is denoted by μ_A,

$$\mu_A : X \to [0,1]. \text{ Hence } \forall x \in X, \mu_A(x) \in [0,1] \text{ is the membership grade of } x.$$

This notation is perhaps more comprehensive since its notation of the fuzzy set, A, from the notation of its membership function μ_A (Zimmermann, 2001).

Secondly, the function is denoted by A and will have the form, $A: X \rightarrow [0,1]$. In this case, the distinction of the fuzzy set and its membership notation is not made (Zimmermann, 2001). However, this does not result in any ambiguity since each fuzzy set is uniquely defined by one specific membership function (Palaniappan, 2002). The choice between these notations is driven solely by convenience and the ease of explanation (Dubois & Prade, 2008; Klir & Yuan, 1995; Zimmerman, 2001).

Example 2.2.1

One of the distinguishing properties between classical (crisp) sets and fuzzy sets, is the fact that representation of fuzzy sets does not only depend on the concept, but also on the context in which it is used, (Dubois & Prade, 2008; Klir & Yuan, 1995; Zimmerman, 2001). For instance, the concept of high temperature in weather science where a 'high temperature' might refer to 35 – 45°C, differs greatly in the context of nuclear science, where a 'high temperature' could mean thousands of degrees Celsius (Klir & Yuan, 1995). Even for similar contexts, fuzzy sets representing the same concept may vary considerably. Nevertheless, the concept must be grounded by similarity in some key features. The following basic definitions in fuzzy set theory are drawn from Zimmermann (2001) and are a few of the foundational building blocks of fuzzy set theory.

2.2.3 Crossover Points, Height, Core and Support

In fuzzy set theory, the concepts of crossover points, height, core, and support are fundamental in characterising the structure and behavior of membership functions of a fuzzy set (Dubois & Prade, 2008; Klir & Yuan, 1995; Zimmerman, 2001). As noted by Zimmermann (2001), these concepts help in defining the uncertainty, degree of inclusion, and boundary properties of fuzzy sets, which are critical in applications.

The elements of A such that $A(x) = 0.5$ are called the crossover points of A. The height h of a fuzzy set A is the maximum value of the membership function;

$$h(A) = \max\{\mu_A(x)\} \tag{2.1}$$

The core value of a fuzzy set are all values of the set which are characterized by full membership,

$$Core(A) = \{x \in X \,|\, \mu_A(x) = 1\}. \tag{2.2}$$

The support of a fuzzy set A, is the crisp set of all $x \in X$ such that $\mu_A(x) > 0$

$$Support(A) = \{x \in X \,|\, \mu_A(x) > 0\}. \tag{2.3}$$

The boundary of a membership function for a fuzzy set A is defined as that region of universe X, that is characterized by non-zero membership but not complete membership (Dubois & Prade, 2008; Klir & Yuan, 1995; Zimmerman, 2001). Boundaries comprises in X whose membership value is given by $\mu_A(x) \in (0,1)$.

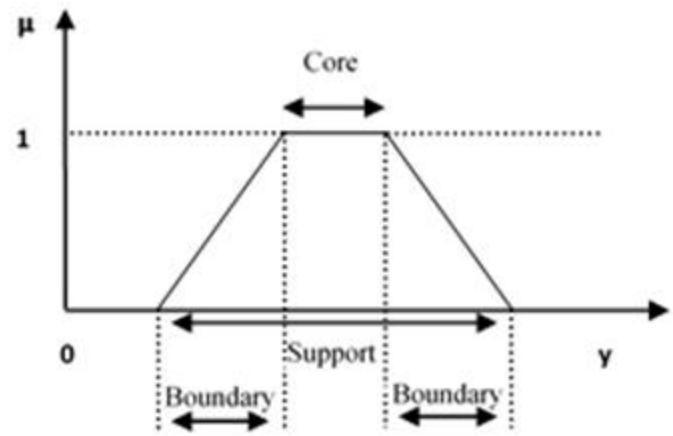

Figure 2.1: Core, support and boundary of a fuzzy set

2.2.4 Normal Fuzzy Set

A fuzzy set is normal if at least one value of the set has full membership i.e., $\sup \mu_A(x) = 1$. A nonempty fuzzy set can be normalized by dividing $\mu_A(x)$ by $\sup \mu_A(x)$. For convenience, and as suggested by Zimmerman (2001) we generally assume that fuzzy sets are normal.[1]

2.2.5 Membership Functions

Membership functions fully define a fuzzy set. Each element of the fuzzy set will be uniquely determined by the membership function. The membership function, according to Zimmerman (2001), will provide a measure of the degree of similarity of an element to the descriptive name of the fuzzy set. Since every fuzzy set is characterised by its membership function, any modification to this function alters the entire definition of the fuzzy set.

[1] A normal fuzzy set has at least one element that fully belongs to the set, providing a clear reference point. Without normality, a fuzzy set would lack a "most representative" element, making interpretation difficult.

Example 2.2.2

Consider the of a class of real numbers close to 2 as provided by (Zimmermann, 2001). The figure below shows some membership functions representing this conception.

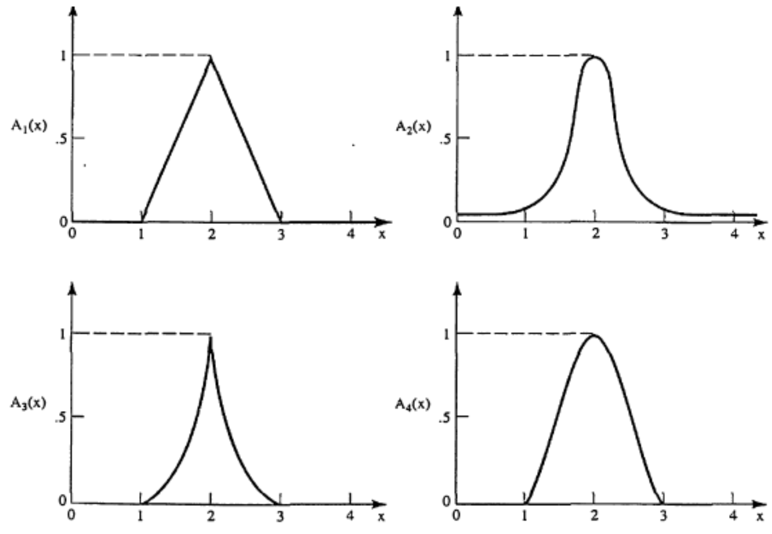

Figure 2.2: Membership functions for real numbers 'close to 2'

Although the functions in Figure 2.2 above are clearly different, the four fuzzy sets are similar since they possess the following properties $\forall A_i : i \in \mathbb{N}_4$.

i) $A_i(2) = 1$ and $A_i(x) < 1 \ \forall x \neq 2$

ii) A_i is symmetric with respect to $x = 2$ i.e., $A_i(2+x) = A_i(2-x), \forall x \in \mathbb{R}$

iii) $A_i(x)$ decreases monotonically from 1 to 0 with increasing difference $|2-x|$

These properties are necessary in order to accurately represent the given idea. It makes sense to require that any additional fuzzy sets representing the same conception must possess these properties (Dubois & Prade, 2008; Klir & Yuan, 1995; Zimmerman, 2001).

Further, referring to Figure 2.2, the given functions exhibit similarity in the way they treat the values that are outside the interval [1,3]. This is primarily because their membership grades are either zero or insignificantly small. This exclusion reflects the fundamental principle of support sets in fuzzy logic, where elements with negligible membership contribute minimally to computations and to application problems (Zadeh, 1965).

Despite differences in graphical representation of fuzzy sets, the suitability of a particular membership function is largely dependent on context (Dubois & Prade, 2008; Klir & Yuan, 1995; Zimmerman, 2001). Appendix B of this thesis provides a summary of frequently used membership functions. It is worth noting that different application problems will work well with different membership function depending on what is required as an output (Zimmermann, 2001). However, in many practical scenarios, small variations in graph shape do not significantly impact the output of the system. This can be explained by the fact that these small deviations will likely lead to small changes in overall decision-making or inference processes (Klir & Yuan, 1995).

Given this flexibility, simpler membership functions are often preferred (Dubois & Prade, 1980). The triangular membership function, for example, is widely used due to its simplicity. This function also provides ease of parameterisation, and the ability to approximate more complex functions effectively (Dubois & Prade, 1980).

2.2.6 Cardinality of a Fuzzy Set

Since elements in a fuzzy set A have degrees of membership in the range [0,1], the cardinality of a fuzzy set must account for these fractional values rather than simply counting elements as in classical sets (Dubois & Prade, 1980).

For a finite fuzzy set A, the scalar cardinality $|A|$ is given by

$$|A| = \sum \mu_A(x) \tag{2.4}$$

The relative cardinality of A is $\|A\| = \frac{|A|}{|X|}$, where $|X|$ is the cardinality of the universal set

In fuzzy sets, the cardinality reflects the aggregate strength of membership across elements. Therefore, if all elements have a membership degree of 1, the fuzzy cardinality equals the classical count (Dubois & Prade, 1980).

Example 2.2.3

Suppose the graph below shows the membership functions "representing the concepts of a young, middle-aged and old person" as discussed by Bellman and Giertz (1973) in their conceptualisation of fuzzy sets

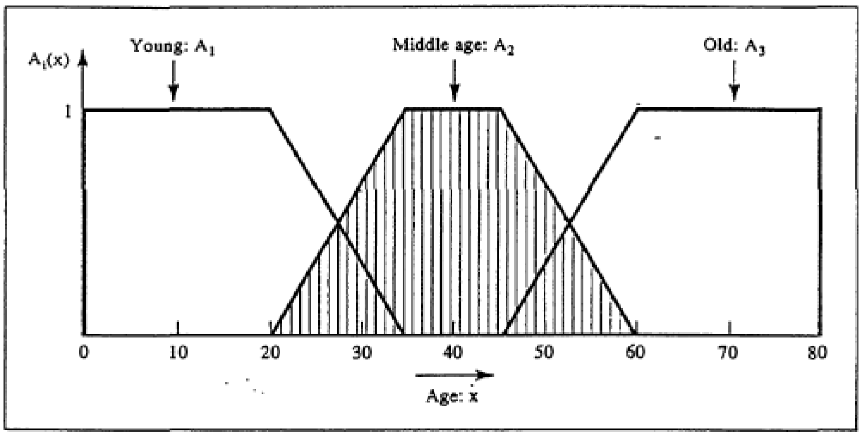

Figure 2.3: Membership functions representing age

The concept of cardinality in fuzzy sets is both fundamental and highly applicable across various disciplines. In decision-making systems, fuzzy cardinality is used to evaluate the extent to which certain conditions are met, for instance when assessing risk levels in finance or determining the severity of medical conditions in healthcare (Zimmermann, 2001). In image processing, fuzzy cardinality helps in feature extraction and segmentation by measuring the intensity or presence of certain patterns, making it useful in facial recognition, medical imaging, and object detection (Chamorro-Martínez et al., 2014).

Additionally, in environmental monitoring, fuzzy cardinality can be used to assess the quality of air as well as in modelling climatic conditions (Dubois & Prade, 2012). By allowing for gradual membership rather than rigid classifications, fuzzy cardinality provides a powerful mathematical tool that can handle imprecision and data that is subjective. This capability makes it highly relevant in domains requiring flexible, human-like reasoning (Klir & Yuan, 1995).

2.2.7 Fuzzy Set Operations

The extension of set theoretic operations to fuzzy set theory was originally proposed by Zadeh 1965. The following definitions have been adapted by different researcher (Bellman & Giertz, 1973; Palaniappan, 2002; Zimmerman 1999).

The definition below was formulated by Zimmerman (2001) to define operations under fuzzy sets

Let $A, B \in I^X$ be two fuzzy sets, then

i) Union: $\left(A\vee B\right)\left(x\right)=\max\left\{A(x),B(x),\forall x\in X\right\}$

ii) Intersection: $\left(A\wedge B\right)\left(x\right)=\min\left\{A(x),B(x),\forall x\in X\right\}$

iii) Complement: $A^{c}(x)=1-A(x)\forall x\in X$

iv) Let $f:X\to Y, A\in I^{X}, B\in I^{Y}$, then $f(A)$ is a fuzzy set in Y defined by

$$f(A)(y)=\begin{cases}\sup\left\{A(x):x\in f^{-1}(y)\right\}, & \text{if } f^{-1}(y)\neq\varnothing\\ \\ 0, & \text{if } f^{-1}(y)=\varnothing\end{cases} \tag{2.5}$$

Example 2.2.4

As demonstrated by Zimmerman (2001), the following example presents the operations of fuzzy sets more clearly. Suppose a realtor wants to classify the house he offers to his clients. One indicator of comfort is the number of bedrooms in the house. Let $X=\{1,2,3,...,10\}$ be the set of the number of bedrooms in a house, A be the fuzzy set of 'comfortable type of house for a family of 4' and B be the fuzzy set of 'large type of house' defined by

$A=\{(1,.2),(2,.5),(3,.8),(4,1),(5,.7),(6,.3)\}$ and

$B=\{(3,.2),(4,.4),(5,.6),(6,.8),(7,1),(8,1)\}$ respectively. Then

i)
$$\begin{aligned}\left(A\vee B\right)\left(x\right)&=\max\left\{A(x),B(x),\forall x\in X\right\}\\&=\{(1,.2),(2,.5),(3,.8),(4,1),(5,.7),(6,.8)(7,1),(8,1)\}\end{aligned} \tag{2.6}$$

ii)
$$\begin{aligned}\left(A\wedge B\right)\left(x\right)&=\min\left\{A(x),B(x),\forall x\in X\right\}\\&=\{(3,.2),(4,.4),(5,.6),(6,.3)\}\end{aligned} \tag{2.7}$$

iii)
$$\begin{aligned}A^{c}(x)&=1-A(x)\forall x\in X\\&=\{(1,.8),(2,.5),(3,.2),(5,.3),(6,.7),(7,1),(8,1),(9,1),(10,1)\}\end{aligned} \tag{2.8}$$

The intersection of fuzzy sets is commonly interpreted as the logical "and". It is then modeled using the minimum (min) operator. On the other hand, the union is understood as the logical "or" and is represented by the maximum (max) operator. Over time, researchers have introduced and refined additional operators to enhance the flexibility of fuzzy set operations. Among these, the most widely used are the triangular norms (t-norms) and triangular conorms (t-conorms). These serve as generalised operators for intersection and union, respectively (Zimmermann, 2001).

2.2.8 Triangular Norm and Triangular co-norm

The foundation for t – norms was first proposed by Menger (1942) in the context of constructing metric spaces. In this case probability distributions as opposed to numerical values are used to describe distances between elements in a given space. However, it was Schweizer and Sklar (1961) who formally axiomatised the concept of t – norms, providing a structured mathematical framework for their application in fuzzy logic and metric spaces (Schweizer & Sklar, 1961). Since then, t – norms and t – conorms have been extensively studied and applied in fuzzy reasoning, decision-making systems, and uncertainty modeling, making them fundamental tools in fuzzy set theory.

A triangular norm (t – norm) is a binary operation T on the unit interval $[0,1]$ i.e., $T:[0,1]^2 \rightarrow [0,1]$ such that $\forall x, y, z \in [0,1]$ (Zimmerman, 2001)

$$\begin{aligned}
&\text{Commutativity:} && T_1 : T(x,y) = T(y,x) \\
&\text{Associativity:} && T_2 : T(T(x,y),z) = T(x,T(y,z)) \\
&\text{Monotonicity:} && T_3 : T(x,y) \le T(x,z), \text{ whenever } y \le z \\
&\text{Boundary Condition:} && T_4 : T(x,1) = x
\end{aligned} \tag{2.9}$$

The function T define a general class of intersection operators for fuzzy sets. Since the operators are associative, then according to Zimmerman (2001), it is possible to compute membership values for the intersection of more than two fuzzy sets by applying the t – norm operator recursively.

There exists uncountably many t – norms. The following is a list of four basic t – norms as defined by Cao et al., (1986).

i. Minimum: $T_M : T_M(x,y) = \min(x,y)$

ii. Maximum: $T_P : T_P(x,y) = x \cdot y$

iii. Lukasiewicz: $T_L = \max(x + y - 1, 0)$ (2.10)

iv. Drastic product: $T_D : T_D(x,y) = \begin{cases} 0, & (x,y) \in [0,1]^2 \\ \min(x,y), & otherwise \end{cases}$

While the $t-$norm defines a general class of intersection operators for fuzzy sets, the triangular conorm ($t-$conorm) defines a general class of aggregation operators for the union of fuzzy sets in an analogous manner (Zimmerman, 2001).

A triangular conorm ($t-$ conorm) is a binary operation S on the unit interval $[0,1]$ i.e, $S:[0,1]^2 \to [0,1]$ such that $\forall x,y,z \in [0,1]$

$$\begin{aligned} &\text{Commutativity:} && S_1 : S(x,y) = S(y,x) \\ &\text{Associativity:} && S_2 : S(S(x,y),z) = S(x,S(y,z)) \\ &\text{Monotonicity:} && S_3 : T(x,y) \le S(x,z), \text{ whenever } y \le z \\ &\text{Boundary Condition:} && S_4 : S(x,0) = x \end{aligned} \tag{2.11}$$

The following is a list of four basic $t-$conorms (Cao et al., 1986)

i. Maximum: $S_M : S_M(x,y) = \max(x,y)$

ii. Probabilistic sum : $S_P : S_P(x,y) = x + y - x \cdot y$

iii. Lukasiewicz: $S_L = \min(x+y,1)$ (2.12)

iv. Drastic product: $S_D : S_D(x,y) = \begin{cases} 1, & (x,y) \in [0,1]^2 \\ \max(x,y), & otherwise \end{cases}$

Clearly, $t-$norms and $t-$conforms are related in a sense of a logical duality. It is possible to define $t-$conorm from the viewpoint of the $t-$norm. Thus, a $t-$conorm is a function $S:[0,1]^2 \to [0,1]$ such that the function T defined by $T(x,y) = 1 - S(1-x,1-y)$ is a $t-$ norm. The pairs, $(T_M, S_M), (T_P, S_P), (T_D, S_D), (T_L, S_L)$, of $t-$norms and $t-$conforms are dual to each other.

Example 2.2.5

t – norms find application in fuzzy control systems, which are control systems that are based on fuzzy logic (Dubois & Prade, 1980; Klement et al., 2001). Fuzzy control systems consist of linguistic rules whose interpretation, in the composition and inference phases, involve the use of t – norms and t – conorms for modelling intersection and union operation of fuzzy sets (Klement et al., 2001). For example, in temperature control systems, linguistic rules such as:

"IF the temperature is high AND the humidity is low, THEN decrease the fan speed"

would require a t – norm to evaluate the intersection between "high temperature" and "low humidity," determining the overall truth value of the antecedent condition (Mendel, 2001). Conversely, in risk assessment models, a t – conorm might be used to evaluate statements like:

"IF the investment risk is high OR the market volatility is high, THEN avoid the stock"

would require a t – conorm to model the conditions between 'investment risk is high' and 'market volatility is high' ensuring that the decision-making process appropriately accounts for both factors (Dubois & Prade, 2012). The max operator is commonly used in this context, as it ensures that if either investment risk or market volatility is significantly high, the system strongly recommends avoiding the stock. According to Yager, (1980), more complex t – conorms, for instance, the probabilistic sum, can be used to model any nonlinear dependencies. Such operators will be more valuable in models that have been designed for use in credit scoring or even portfolio optimisation.

Another example of a practical problem whose solution can be formulated using fuzzy sets, has been provided in Appendix C of this thesis. This real-world application of fuzzy sets has been included in this thesis to bridge the gap between what is formulated in an abstract way and how such results can be used in practice.

2.3 Fuzzy Metric Space

How to define a fuzzy metric space is a question that remain to be very important in fuzzy mathematics (Dubois & Prade, 1980; Klement et al., 2001). This is because a fuzzy metric space has extensive applications in many real-world situations. For instance, in fuzzy optimisation, where decision-making processes must handle uncertainty, and in pattern recognition, where fuzzy metrics help to classify and compare data patterns (Slapal, 2013). Given its significance, researchers continue to investigate more refined and useful fuzzy measures that can address existing limitations and enhance mathematical modeling in complex systems (Dubois & Prade, 1980; Klement et al., 2001). These measures have been seen to improve mathematical accuracy during computations as well as in problems involving fuzzy reasoning. By so doing, it was noted by Khan (2017), that fuzzy metrics can make problems in artificial intelligence much simpler and therefore expand the scope of applications of AI.

In traditional metrics like the Euclidean or Hamming distances, objects are classified as either identical or completely different (Slapal, 2013). They are identical if the distance between them is zero and different of the distance between them is 1 (Krentel, 1988). However, as already discussed in the proceeding sections, scenarios in real life are rarely binary, since rather than falling into strict categories of identical or distinct objects may share partial similarities or gradual variations (Dubois & Prade, 1980; Klement et al., 2001).

Fuzzy metrics address this complexity by introducing a continuous similarity scale between 0 and 1. (Dubois & Prade, 2012). Unlike the usual distance measures, which provide rigid separations, fuzzy metrics allow for gradual transitions between similarity levels, making them highly effective in dealing with subjective classifications (Zimmermann, 2001). This section discusses the concept of metric spaces and other related concepts in the realm of fuzzy sets.

Definition 2.3.1 (Fuzzy Metric)

A fuzzy metric, also called a fuzzy distance or fuzzy similarity measure, is a concept used in fuzzy logic and fuzzy set theory to quantify the similarity or dissimilarity between two fuzzy sets by considering the degrees of membership of their elements (Dubois & Prade, 1980; Klement et al., 2001). In this way, a means of measuring the ‘closeness’ between fuzzy sets is provided.

Some of the different types of fuzzy metrics that have already been developed include the Hamming metric, which measures differences in membership values, and the Euclidean metric, which considers geometric distance between fuzzy sets (Dubois & Prade, 1980; Klement et al., 2001). Another metric is the Jaccard metric which evaluates the ratio of shared to total memberships, while the cosine metric measures the angular similarity of membership distributions.

Additionally, the Minkowski metric generalises distance in order to provide more adaptable similarity measurements (Dubois & Prade, 2012; Klement et al., 2001).

2.3.1 Characterisation of Fuzzy Metrics using Fuzzy Scalars

Suppose (x,λ) and (y,γ) are two fuzzy scalars, then

i) $(x,\lambda)\succeq(y,\gamma)$ if $x>y$ or $(x,\lambda)=(y,\gamma)$

ii) (x,λ) is said to be no less than (y,γ) if $x\geq y$, denoted by $(x,\lambda)\succ(y,\gamma)$ or $(y,\lambda)\prec(x,\gamma)$

iii) (x,λ) is said to be nonnegative if $x\geq 0$. The set of all nonnegative scalars is denoted $S_F^{+}(\mathbb{R})$

Let X be a nonempty set and $d_F : P_F(X)\times P_F(X)\rightarrow S_F^{+}(\mathbb{R})$ be a mapping. $(P_F(X),d_F)$ is called a fuzzy metric space if for any $\{(x,\lambda),(y,\gamma),(z,\delta)\}\subset P_F(X)$ then d_F satisfies the following conditions;

i) Nonnegative: $d_F((x,\lambda),(y,\gamma))=0$ *iff* $x=y$ and $\lambda=\gamma=1$

ii) Symmetric: $d_F((x,\lambda),(y,\gamma))=d_F((y,\gamma),(x,\lambda))$ (2.13)

iii) Triangle inequality: $d_F((x,\lambda),(z,\delta))\prec d_F((x,\lambda),(y,\gamma))+d_F((y,\gamma),(z,\delta))$

d_F is called a fuzzy metric defined in $P_F(X)$ and $d_F((x,\lambda),(y,\gamma))$ is called a fuzzy distance between the two fuzzy points. This approach was postulated by both Anage & Salunke, (2010) and Xia & Guo, (2004). A different version of the definition had also been stated earlier by George & Veeramani, (1994).

An alternative definition of the fuzzy metric space was proposed by Kramosil and Michalek (1975) and further enhanced by many researchers, including Kaleva and Seikkala (1984) and George and Veeramani (1994). In particular, Kramosil and Michalek (1975) defined Hausdorff topology of metric spaces, which was later proved to be metrisable. They also showed that every metric induces a fuzzy metric.

2.3.2 Characterisation of Fuzzy Metric using *t* - norm

The triple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t – norm and M is a fuzzy set on $X^2 \times [0, \infty)$ such that $\forall x, y, z \in X$ and $s, t > 0$, the following conditions are satisfied;

i) $M(x, y, 0) = 0$

ii) $M(x, y, t) = 1 \;\; iff \;\; x = y$

iii) $M(x, y, t) = M(y, x, t)$ (2.14)

iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$

v) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous

$M(x, y, t)$ is interpreted as the degree of nearness of x and y with respect to t (Zimmermann, 2001).

2.3.3 Induced Fuzzy Metrics

Every metric naturally defines a corresponding fuzzy metric. As demonstrated by Zimmermann (2001), the following two examples illustrate this transformation

Example 2.3.1

Let (X, d) be a metric space on which the $a * b = ab \;\; \forall a, b \in [0, 1]$. Let M_d be a fuzzy set defined on $X^2 \times [0, \infty)$ by $M_d(x, y, t) = \dfrac{t}{t + d(x, y)}$. The triple (X, M_d, t) is a fuzzy metric called the intuitionistic fuzzy metric or the standard fuzzy metric induced by the usual metric d

Indeed,

i) $$M_d(x, y, t) = \frac{t}{t + d(x, y)} > 0 \qquad (2.15)$$

ii) Suppose $M_d(x,y,t)=\dfrac{t}{t+d(x,y)}=1$ then $\dfrac{t}{t+d(x,y)}=1$ and

$$d(x,y)=0 \Rightarrow x-y=0 \Rightarrow x=y \tag{2.16}$$

Suppose $x=y$, then $d(x,y)=0 \Rightarrow M_d(x,y,t)=\dfrac{t}{t}=1$

$M_d(x,y,t)=1 \; iff \; x=y$

iii) $$M_d(x,y,t)=\frac{t}{t+d(x,y)}=\frac{t}{t+d(y,x)}=M_d(y,x,t) \tag{2.17}$$

iv) $M_d(x,z,t+s)\le M_d(x,y,t)+M_d(y,z,s)$ since $d(x,z)\le d(x,y)+d(y,z)$

Therefore $$\frac{t}{t+d(x,z)}\le\frac{t}{t+d(x,y)}+\frac{t}{t+d(y,z)} \tag{2.18}$$

This standard fuzzy metric can be generalized to $M_d(x,y,t)=\dfrac{kt^n}{kt^n+md(x,y)}$ where $k,m,n\in\mathbb{R}^+$

This example would still hold if the t – norm defined above was substituted by . $a*b=\min(a,b)$

Example 2.3.2

Let $X=\mathbb{R}.$ Define $a*b=ab$ and $M(x,y,t)=\left(\exp\left(\dfrac{|x-y|}{t}\right)\right)^{-1}$ $\forall x,y\in X, t>0.$ Then the triple $(X,M,*)$ defines a fuzzy metric

Indeed,

i) Nonnegative: $$M(x,y,t)=\left(\exp\left(\frac{|x-y|}{t}\right)\right)^{-1}>0 \tag{2.19}$$

ii) Zero-distance: Suppose $M(x,y,t)=\left(\exp\left(\dfrac{|x-y|}{t}\right)\right)^{-1}=1$ then $\dfrac{1}{\left(\exp\left(\dfrac{|x-y|}{t}\right)\right)}=1$ and

$$\frac{|x-y|}{t}=0 \text{ hence } |x-y|=0 \Rightarrow x-y=0 \Rightarrow x=y \tag{2.20}$$

On the other hand if $x = y$ then $M(x, y, t) = \exp(0) = 1$

Hence $M(x, y, t) = 1$ *iff* $x = y$

iii) Symmetry: $M(x, y, t) = \left(\exp\left(\frac{|x-y|}{t} \right) \right)^{-1} = \left(\exp\left(\frac{|y-x|}{t} \right) \right)^{-1} = M(y, x, t)$ (2.21)

iv) Triangle inequality:

$$M(x, y, t) M(y, z, s) \le M(x, z, t+s)$$
$$|x-z| \le \frac{t+s}{t}|x-y| + \frac{t+s}{s}|y-z|$$
$$\frac{|x-z|}{t+s} \le \frac{|x-y|}{t} + \frac{|y-z|}{s}$$
$$\exp\left(\frac{|x-z|}{t+s} \right) \le \exp\left(\frac{|x-y|}{t} \right) \exp\left(\frac{|y-z|}{s} \right) \tag{2.22}$$
$$\left[\exp\left(\frac{|x-z|}{t+s} \right) \right]^{-1} \le \left[\exp\left(\frac{|x-y|}{t} \right) \exp\left(\frac{|y-z|}{s} \right) \right]^{-1}$$
$$\left[\exp\left(\frac{|x-z|}{t+s} \right) \right]^{-1} \ge \left[\exp\left(\frac{|x-y|}{t} \right) \right]^{-1} \left[\exp\left(\frac{|y-z|}{s} \right) \right]^{-1}$$
$$M(x, z, t+s) \ge M(x, y, t) M(y, z, s)$$

v) $M(x, y, \cdot) : (0, \infty) \to [0,1]$ is continuous

$(X, M, *)$ is a Fuzzy Metric Space

2.4 Entropy and Similarity Measures

Several researchers have noted that a fundamental question in the study of fuzzy sets in has to do with the use of relative comparisons to calculate the similarities and distances between fuzzy sets" (Dubois & Prade, 2012; Chaudhuri & Rosenfeld, 1996; Zadeh, 1965). Similarity involves recognizing patterns and making associations which enable one to classify objects and concepts, particularly in the domain of classification and clustering. In these applications, an unknown object is assigned to a particular category if its similarity measure to objects within that category is higher than its similarity to objects in other categories (Zimmermann, 2001).

2.4.1 Similarity

Intuitively, the similarity of two fuzzy sets is 1 if they are identical i.e., if they both contain the same values with the same degree of membership, and 0 if they have nothing in common, i.e., they do not contain any of the same values. These two properties are referred to as reflexivity and overlapping respectively.

Definition 2.4.1

A function $S: A \times B \rightarrow [0,1]$ is called a similarity measure if

i) Reflexivity: $S(A, A) = 1$, meaning an element always has full similarity to itself
ii) Symmetry: $S(A, B) = S(B, A) \quad \forall A, B \in X$. This property ensures that similarity is bidirectional
iii) Overlapping: $S(A, B) \geq \max\left(\frac{|A \cap B|}{|A|}, \frac{|A \cap B|}{|B|}\right)$. This property ensures that shared characteristics contribute strongly to the similarity score
iv) Transitivity: if $S(A, B) \geq \alpha$ and $S(B, C) \geq \alpha$ then $S(A, C) \geq \alpha$. This property is important for clustering and classification, ensuring that if A is similar to B and B is similar to C, then A should be similar to C.

It is not necessary for a similarity measure to have all these properties Tversky (1977). There are situations when symmetry does not need to be satisfied and in other situations, it has been argued whether transitivity is necessary or even useful. In his model, Tversky (1977) suggested that human judgments of similarity often violate these properties in two primary ways.

i) **Symmetry Violations**: People might judge North Korea to be more similar to China than China is to North Korea, reflecting the influence of a reference point or salience in comparisons. This asymmetry arises because people tend to use a more familiar or prominent object as a reference point and compare the less familiar object to it. Since China is a large, well-known country while North Korea is smaller and more isolated, people judge North Korea to be similar to China, but not vice versa.

ii) **Triangle Inequality Violations**: Tversky highlighted that if Jamaica is similar to Cuba (both are Caribbean islands) and Cuba is similar to Russia (both have a history of communism), it doesn't necessarily imply that Jamaica is similar to Russia, thereby questioning the applicability of the triangle inequality in similarity judgments.

Findings from Tversky (1977) imply that human similarity judgments are context-dependent, asymmetric, and often violate transitivity hence challenging traditional metric-based models. Unlike mathematical similarity measures that assume symmetry and transitivity, human perception is very highly subjective. This means that classical metric-based similarity models may fail to capture the flexibility of human cognition.

2.4.2 The Jaccard Similarity

The Jaccard similarity index, as noted by Tversky (1977), measures the similarity between two fuzzy sets by computing the ratio of their intersection to their union, quantifying the degree of overlap between them. According to Dubois and Prade (1980), the fuzzy Jaccard similarity between two fuzzy sets A and B in a universal set X is defined as,

$$J\left(A,B\right)=\frac{\sum_{x\in X}\min\left(\mu_A\left(x\right),\mu_B\left(x\right)\right)}{\sum_{x\in X}\max\left(\mu_A\left(x\right),\mu_B\left(x\right)\right)} \tag{2.23}$$

Instead of simply counting shared elements, fuzzy Jaccard similarity is computed using the ratio of the sum of the minimum membership values to the sum of the maximum membership values (Tversky, 1977).

If two fuzzy sets are highly similar (i.e., Jaccard similarity is high), their combined uncertainty or fuzziness is reduced because they share a large proportion of elements (Tversky, 1977). Conversely, if two fuzzy sets have a low Jaccard similarity, they have more distinct characteristics, leading to greater overall entropy when considering both sets.

2.5 Entropy

Entropy, a fundamental concept in information theory, measures the uncertainty or randomness of a system. Shannon entropy, introduced in 1948, is the most well-known measure of information uncertainty and forms the basis of modern information theory (Shannon, 1948). However, Shannon entropy has limitations when dealing with higher-order moments of probability distributions, long-tailed distributions, or multi-fractal systems. In such scenarios, a more generalized entropy measure is required (Shannon, 1948).

To address these limitations, Alfréd Rényi introduced the Rényi entropy in 1961 as a one-parameter generalisation of Shannon entropy (Rényi, 1961). Rényi entropy provides flexibility in quantifying information by introducing an order parameter, α. This order parameter allows researchers to fine-tune the sensitivity of models to different probabilities within a distribution.

2.5.1 Classical Shannon and Rényi Entropies

The Rényi entropy of order α for a probability distribution $P = \{p_1, p_2, ..., p_n\}$ is defined as

$H_\alpha(P) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{n} p_i^\alpha$ where $\alpha > 0, \alpha \neq 1$ is the order of the entropy, controlling the weighting of probabilities

Note

i) When $\alpha \to 1$ Renyi entropy converges to Shannon entropy

$$H_1(P) = \sum_{i=1}^{n} p_i \ln p_i$$

ii) When $\alpha \to \infty$ Renyi entropy focusses only on the most probable event.

2.5.2 Entropy on Fuzzy Sets

In the context of fuzzy sets, the concept of the entropy of a fuzzy set was first introduced by Zadeh (1965). This was mainly driven by a need to quantify the degree of fuzziness within a fuzzy set. As a result, Zadeh’s work laid the foundation for measuring the uncertainty or ambiguity present in fuzzy systems. The formal axiomatisation of fuzzy entropy was later developed by De Luca and Termini (1972), who drew parallels between fuzzy entropy and Shannon probability entropy snd saw this as an opportunity to interpret the concept as a measure of the amount of information contained in a fuzzy set.

Further advancements were made by Kaufmann (1975), who proposed that the entropy of a fuzzy set could be calculated based on its distance from the nearest crisp (non-fuzzy) set. This formulation provided a different perspective on fuzzy measurement that is geometric in nature.

Later, Yager (1979, 1982) offered an alternative interpretation, defining entropy as the distance between a fuzzy set and its complement. According to Yager (1979), any meaningful measure of fuzziness should reflect the lack of distinction between a fuzzy set and its negation. This interpretation culminated in defining a metric that may be used to measure the distance between a fuzzy set and its complement.

2.5.3 Formulation of Entropy on Fuzzy Sets

Formally, in Zimmermann (2001) the entropy $H(A)$ of a fuzzy set A defined on a universal set X with membership function is expressed as:

$$H(A) = -\sum_{x\in X}\left[\mu_A(x)\log\mu_A(x) + \left(1-\mu_A(x)\right)\log\left(1-\mu_A(x)\right)\right] \tag{2.24}$$

where $\mu_A(x)$ represents the membership degree of element x in the fuzzy set A.

Example 2.5.1

Define $d_p\left(A, A^c\right) = \left[\sum_{i=1}^{n}\left|\mu_A(x_i) - \mu_{A^c}(x_i)^p\right|\right]^{\frac{1}{p}}$, $p = 1, 2, 3, \ldots$ Then the measure of fuzziness of A can be defined as

$$f_p(A) = 1 - \frac{d_p\left(A, A^c\right)}{\left\|\sup(A)\right\|} \tag{2.25}$$

For $p = 1$, $d_p\left(A, A^c\right)$ yields the Hamming metric.

$$d_p\left(A, A^c\right) = \sum_{i=1}^{n}\left|\mu_A(x_i) - \mu_{A^c}(x_i)\right| \tag{2.26}$$

Since $\mu_{A^c}(x) = 1 - \mu_A(x)$, this becomes $d_p\left(A, A^c\right) = \sum_{i=1}^{n}\left|2\mu_A(x_i) - 1\right|$

For $p = 2$ we have the Euclidean metric

$$d_2\left(A, A^c\right) = \left[\sum_{i=1}^{n}\left|\mu_A(x_i) - \mu_{A^c}(x_i)^2\right|\right]^{\frac{1}{2}} \text{ and}$$

Since $\mu_{A^c}(x) = 1 - \mu_A(x)$, then this becomes

$$d_2\left(A, A^c\right) = \left[\sum_{i=1}^{n}\left(2\mu_A\left(x_i\right) - 1\right)^2\right]^{\frac{1}{2}} \tag{2.27}$$

2.6 Intuitionistic Fuzzy Sets, IFS

Fuzzy set theory provides an elegant generalization of the crisp sets by allowing elements to have varying degrees of membership and thereby accommodating vagueness and uncertainty in mathematical modeling. In this framework, the degree of non-membership of an element in a fuzzy set is typically represented as $1 - \mu_A(x)$, where $\mu_A(x)$ is the degree of membership of x in A (Dubois & Prade, 2012).

However, as noted by Torra and Narukawa (2009), although this approach is mathematically convenient, it enforces a fixed complementary relationship between membership and non-membership degrees, potentially oversimplifying inherent uncertainty in certain situations. This oversimplification arises because there will always be some element of indecision in assigning accurate membership and non-membership degrees on fuzzy sets, leading to a lack of flexibility in modeling subjective situations (Torra & Narukawa, 2009).

A more satisfactory way of dealing with this hesitation is to describe the degree of non-membership of x in A that is independent of the membership function μ. This generalization was proposed by Atanassov (1983) to incorporate the degree of hesitation called the intuitionistic index.

2.6.1 Mathematical Properties of Intuitionistic Fuzzy Sets

Let $X \neq \varnothing$. An intuitionistic Fuzzy Set A in X is an object having the form $A = \left\{x, \mu_A(x), \varphi_A(x) \middle| x \in X\right\}$ where;

i) $\mu_A(x): X \to [0,1]$ and $\varphi_A(x): X \to [0,1]$ define the degree of membership and nonmembership of x in A respectively (Atanassov, 1983). Of course, $0 \leq \mu_A(x) + \varphi_A(x) \leq 1$.

ii) The value $\pi_A(x) = 1 - \mu_A(x) + \varphi_A(x)$ is called the intuitionistic index or hesitation margin of x in A. Intuitively, $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Example 2.6.1

Let A be an IFS with $\mu_A(x) = 0.5, \varphi_A(x) = 0.3$ then $\pi_A(x) = 1 - (0.5 + 0.3) = 0.2$. This means that the degree that x belongs to IFS A is 0.5, the degree that x does not belong to IFS A is 0.3 and the degree of hesitancy is 0.2.

2.6.2 Applications of IFS

The conception of IFS provides a more flexible means of handling uncertainty thereby providing a more consistent human reasoning for imprecise knowledge. The additional degree of freedom allows for better modelling of imperfect information which is omnipresent in any conscious decision making (Szmidt & Kacprzyk, 2004). Intuitionistic fuzzy sets have found application in the modelling of real-life problems like sale analysis, new product marketing, financial services, negotiation process, psychological investigations etc., since there is a fair chance of the existence of a non-null hesitation part at each moment of evaluation of an unknown object (Szmidt & Kacprzyk, 2002).

IFS have been used to model medical diagnostic reasoning (Szmidt & Kacprzyk, 2004). The proposed method of diagnosis involves measuring the similarity for intuitionistic fuzzy sets, while taking into consideration the fact that for different patients suffering the same illness, values of the same symptoms can be different. An in-depth understanding of this approach can be found in (Szmidt & Kacprzyk, 2002, 2004).

Another application of IFS in career determination was proposed by Ejegwa et al., (2014) using normalized Euclidean distance to assess the proximity of a student's academic performance to a particular career choice. The mathematical model for this proposition has been included in Appendix D of this research. The aim is to further illustrate the utility of IFS and provide a computational example that enhances the understanding of its real-world application.

2.7 Bipolar Fuzzy Sets, BFS

While IFS extends fuzzy sets by introducing an additional parameter to account for hesitation, it still operates within a single-dimensional framework, where membership and non-membership are inherently linked and constrained by the condition $0 \le \mu_A(x) + \varphi_A(x) \le 1$ (Zhang, 1994).

It turns out that most human processes are not just fuzzy (Zhang, 1994). They are instead double-sided. Most phenomena will exhibit both a positive side and a negative side. In general, relations, communication or even feelings among humans have two aspects, the positive and the negative. The positive aspect expresses possible or permitted evidences which provide satisfaction while the negative aspect represents impossible relations or situations which are unacceptable or not permitted and provide dissatisfaction (Atanassov, 1986; Dubois & Prade, 2008). Therefore, it is not enough to have a theory that addresses imprecision, uncertainty and ambiguity; rather, the theory must also be able to model polarity.

A new extension of fuzzy sets has been proposed, which arguably models human experiences more comprehensively (Zhang, 1994). This is the theory of bipolar-valued fuzzy sets, where membership is expressed through two independent satisfaction degrees. Bipolar and intuitionistic fuzzy sets look similar but are essentially different. In many domains, it is necessary to be able to deal with bipolar information – information possessing both what is possible and what is not possible. It is noted that positive information represents what is granted to be possible while negative information represents what is granted to be impossible (Zhang, 1998). According to Zhang, (1998), dual Boolean logic lacks the representational and reasoning capabilities for directly modeling the coexistence and interaction of bipolar relationships. While fuzzy logic may model uncertainty, it is limited when it comes to describing polarity. To resolve these two features, bipolar fuzzy set theory combines both the fuzziness and polarity into a unified model hence providing a theoretical basis for bipolar clustering, conflict resolution and coordination (Zhang, 2016)

2.7.1 Fundamental Concepts of BFS

A few basic definitions are given here below as found in Kim et al., (2018)

Let X be a nonempty set. A pair $\mu = (\mu^+, \mu^-)$ is called a bipolar-valued fuzzy set (or bipolar fuzzy set) in X if $\mu^+ : X \to [0,1]$ and $\mu^- : X \to [-1,0]$ are mappings.

In particular, the bipolar fuzzy empty set (resp. the bipolar fuzzy whole set), denoted by $0_{bp}=\left(0_{bp}^{+},0_{bp}^{-}\right)$ $\left[\text{resp. } 1_{bp}=\left(1_{bp}^{+},1_{bp}^{-}\right)\right]$, is a bipolar fuzzy set in X defined by;

$$\forall x\in X,\;\; 0_{bp}^{+}(x)=0=0_{bp}^{-}(x)\;\left[\text{resp. } 1_{bp}^{+}(x)=1 \text{ and } 1_{bp}^{-}(x)=-1\right]$$

The collection of all bipolar fuzzy sets of the set X will be denoted by $BPF(X)$

The positive membership degree $\mu^{+}(x)$ denotes the satisfaction degree of the element x to the property corresponding to the bipolar fuzzy set μ while the negative membership degree $\mu^{-}(x)$ denotes the satisfaction degree of x to some implicit counter-property corresponding to the bipolar fuzzy set μ (Zhang, 2016).

Therefore, if $\mu^{+}(x)\neq 0$ and $\mu^{-}(x)=0$, then it is the situation that x is regarded as having only positive satisfaction for μ. On the other hand, if $\mu^{+}(x)=0$ and $\mu^{-}(x)\neq 0$, then it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter-property of μ. If $\mu^{+}(x)\neq 0$ and $\mu^{-}(x)\neq 0$, for some $x\in X$, then the membership function of the property overlaps that of its counter-property over some portion of X.

2.7.2 Operations on BFS

Suppose $X\neq\varnothing$ and $A,B\in BPF(X)$. Then

i) $A\subset B$ implies $A^{+}(x)\leq B^{+}(x)$ and $A^{-}(x)\geq B^{-}(x)$ $\forall x\in X$

ii) The intersection, union and complement of A and B are defined by

$$\left(A\cap B\right)\left(x\right)=\left(A^{+}(x)\wedge B^{+}(x), A^{-}(x)\vee B^{-}(x)\right)$$
$$\left(A\cup B\right)\left(x\right)=\left(A^{+}(x)\vee B^{+}(x), A^{-}(x)\wedge B^{-}(x)\right)$$
$$A^{c}=\left(\left(A^{c}\right)^{+},\left(A^{c}\right)^{-}\right) \text{ where } \left(A^{c}\right)^{+}(x)=1-A^{+}(x) \text{ and } \left(A^{c}\right)^{-}(x)=-1-A^{-}(x) \tag{2.28}$$

iii) In general, if $\left(A_{i}\right)_{i\in I}\subset BPF(X)$ is a collection of bipolar fuzzy subsets of X, then $\forall x\in X$

$$
\begin{aligned}
\left(\bigcap_{i\in I} A_i\right)(x) &= \left(\bigwedge_{i\in I} A_i^+(x), \bigvee_{i\in I} A_i^-(x)\right) \\
\left(\bigcup_{i\in I} A_i\right)(x) &= \left(\bigvee_{i\in I} A_i^+(x), \bigwedge_{i\in I} A_i^-(x)\right)
\end{aligned}
\tag{2.29}
$$

The Boolean laws of s*et al*gebra can be generalized into the context of bipolar fuzzy sets

2.7.3 Mappings on Bipolar Fuzzy Sets

Let $A_X \in BPF(X), A_Y \in BPF(Y)$ and let $f : X \to Y$ be a mapping. Then

i) The image of A_X under f, denoted $f(A_X) = \left(f(A_X^+), f(A_X^-)\right)$ is a bipolar fuzzy set in Y defined by

$$
\left[\left(f\left(A_X^+\right)\right)\right](y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} A_X^+(x), & f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}
\tag{2.30}
$$

$$
\left[\left(f\left(A_X^-\right)\right)\right](y) = \begin{cases} \bigwedge_{x\in f^{-1}(y)} A_X^-(x), & f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}
$$

ii) The preimage of A_Y under f, denoted $f^{-1}(A_Y) = \left(f^{-1}(A_Y^+), f^{-1}(A_Y^-)\right)$ is a bipolar fuzzy set in X defined by

$$
\begin{aligned}
\left[f^{-1}\left(A_Y^+\right)\right](x) &= A_Y^+ \circ f(x) \\
\left[f^{-1}\left(A_Y^-\right)\right](x) &= A_Y^- \circ f(x)
\end{aligned}
\tag{2.31}
$$

2.7.4 Bipolar Fuzzy Point

Let $x \in X$, $(\alpha, \beta) \in [0,1] \times [-1,0]$ and $A \in BPF(X)$. Then

i) $x_{(\alpha,\beta)}$ is called a bipolar fuzzy point in X with value (α,β) and support x if $\forall y \in X$

$$
\left[x_{(\alpha,\beta)}\right](y) = \begin{cases} (\alpha,\beta) & \text{if } y = x \\ (0,0) & \text{otherwise} \end{cases}
\tag{2.32}
$$

ii) $x_{(\alpha,\beta)} \in A$ if

$$
A^+(x) \geq \alpha, \quad A^-(x) \leq \beta
$$

The set of all bipolar fuzzy points in X will be denoted by $BPF_p(X)$. Clearly, every bipolar fuzzy set A can be expressed as a union of all its bipolar fuzzy points.

$$A = \bigcup \left\{ x_{(\alpha,\beta)} \in BPF_P(X) : x_{(\alpha,\beta)} \in A \ \forall BPF\left(X\right) \right\} \quad (2.33)$$

2.7.5 Bipolar Fuzzy Topology

Let $X \neq \varnothing$, and τ be a family of bipolar fuzzy subsets of X. τ is called a bipolar fuzzy topology on X *iff*

$$\begin{aligned} &\text{i)} \quad 0_{bp}, 1_{bp} \in \tau \\ &\text{ii)} \quad A \cap B \in \tau, \text{ whenever } A, B \in \tau \\ &\text{iii)} \quad \bigcup_{i \in I} A_i \in \tau, \ \forall A_i \in \tau, \ i = 1, 2, 3... \end{aligned} \quad (2.34)$$

The pair $\left(X, \tau\right)$ is called a Bipolar Fuzzy Topological Space, $BPFTS$. The members of τ are called $BPFOS$ in X.

$A \in BPF\left(X\right)$ is closed in X if $A^c \in \tau$. The set of all bipolar fuzzy topologies on X is denoted $BPFT\left(X\right)$

2.7.6 Neighborhoods

Let $X \neq \varnothing$ and $\left(X, \tau\right)$ be a $BPFTS$ and $A \in BPF\left(X\right)$. A is called a neighborhood of $x_{(\alpha,\beta)} \in BPF_P\left(X\right)$ if $U \in \tau \ s.t \ x_{(\alpha,\beta)} \in U \subset A$

The set of all bipolar fuzzy neighborhoods of $x_{(\alpha,\beta)}$ is denoted by $\aleph_{bp}\left(x_{(\alpha,\beta)}\right)$

Let $\left(X, \tau_1\right)$ and $\left(Y, \tau_2\right)$ be two bipolar fuzzy topological spaces. Then a mapping $f : X \rightarrow Y$ is continuous at $x_{(\alpha,\beta)} \in BPF_P\left(X\right)$ if

$$\forall V \in \aleph_{bp}\left(f\left(x_{(\alpha,\beta)}\right)\right) = \aleph_{bp}\left(f\left(x\right)_{(\alpha,\beta)}\right) \text{ then } f^{-1}\left(V\right) \in \aleph_{bp}\left(x_{(\alpha,\beta)}\right) \quad (2.35)$$

Equivalently, f is bipolar fuzzy continuous if $\forall \ U \in \tau_2, \ f^{-1}\left(U\right) \in \tau_1$ (resp. $\forall \ V$ closed in $Y, \ f^{-1}\left(V\right)$ is closed in X)

f is called bipolar fuzzy open (resp. bipolar fuzzy closed) if $f\left(A\right)$ is bipolar fuzzy open (resp. bipolar fuzzy closed) whenever A is bipolar fuzzy open (resp. bipolar fuzzy closed)

Theorem

Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be bipolar fuzzy mappings. $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined by

$$(f_1 \times f_2)\left(A_{X_1}, A_{X_2}\right) = \left(f_1\left(A_{X_1}\right), f_2\left(A_{X_2}\right)\right)$$ where

$$\begin{aligned} f_1\left(A_{X_1}\right) &= \left(f_1\left(A_{X_1}^+\right), f_1\left(A_{X_1}^-\right)\right) \\ f_2\left(A_{X_2}\right) &= \left(f_2\left(A_{X_2}^+\right), f_2\left(A_{X_2}^-\right)\right) \end{aligned}$$ are bipolar fuzzy sets given by

$$f_1\left(A_{X_1}^+\right) = \begin{cases} \vee_{x\in f_1^{-1}(y)} A_{X_1}^+(x), & f_1^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases} \tag{2.36}$$

$$f_1\left(A_{X_1}^-\right) = \begin{cases} \wedge_{x\in f_1^{-1}(y)} A_{X_1}^-(x), & f_1^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

If f_1 and f_2 are bipolar fuzzy continuous, then so is the product $f_1 \times f_2$

2.7.7 Bipolar Fuzzy Relation

Let $X, Y \neq \varnothing$ and δ and μ be bipolar fuzzy subsets of X and Y respectively. According to Lee & Hur (2019), a bipolar fuzzy subset $\tilde{R} = \left(\tilde{R}^+, \tilde{R}^-\right) \subset X \times Y$ is called a bipolar fuzzy relation from X to Y if

$$\begin{aligned} \tilde{R}^+(x,y) &\le \min\left\{\delta^+(x), \mu^+(y)\right\} \\ \tilde{R}^-(x,y) &\le \max\left\{\delta^-(x), \mu^-(y)\right\} \quad \forall x \in \delta, y \in \mu \end{aligned} \tag{2.37}$$

2.7.8 Bipolar Fuzzy Graph

According to Akram & Dudek, (2012), a bipolar fuzzy graph with X as the underlying set is a pair $G = \left(\mu, \tilde{R}\right)$ such that $\mu : X \to [-1.1]$ is a bipolar fuzzy subset of X and $\tilde{R} : X \times X \to [-1,1]$ is a bipolar fuzzy relation on A i.e. $\forall x, y \in X$

$$\begin{aligned} \tilde{R}^+(x,y) &\le \min\left\{\mu^+(x), \mu^+(y)\right\} \\ \tilde{R}^-(x,y) &\le \max\left\{\mu^-(x), \mu^-(y)\right\} \end{aligned} \tag{2.38}$$

μ is called the bipolar fuzzy vertex set of G and $\tilde{R}$ the bipolar fuzzy edge set of G. For the definition to make sense, we assume the underlying set X is finite.

2.7.9 Path and Connectedness

A path ρ in a fuzzy graph (A, R) is a sequence of distinct vertices $x_0, x_1, ..., x_n$ such that $R(x_{i-1}, x_i) > 0\ \forall i = 1, 2, ..., n$. $n \geq 1$ is called the length of the path (Zimmermann, 2001).

The introduction of bipolar-valued fuzzy sets marks a significant advancement in fuzzy set theory, allowing for independent and context-sensitive modeling of positive and negative evaluations. This makes BVFS particularly powerful in applications where opposing forces coexist, such as risk assessment, psychology, sentiment analysis, and multi-criteria decision-making. By overcoming the inherent constraints of intuitionistic fuzzy sets, BVFS provides a more flexible and realistic mathematical framework for handling complex, real-world uncertainties (Gorzalczany, 2018).

2.8 Digital Topology

"Digital topology has been developed to address problems in image processing and analysis – an area of computer science that deals with the analysis and manipulation of pictures by computer" (Edward & Lang, 2000). Unlike classical topology, which deals with continuous spaces, digital topology focuses on how pixels or voxels in the digital plane can be arranged to form a discrete lattice.

The primary goal of digital topology is to establish a mathematical frameworks that allows for the analysis of properties of digital spaces with as much rigour as possible. These properties include adjacency relations, connectedness and surroundness. Early developments in digital topology were initiated by Rosenfeld (1979), Khalimsky (1971), and Kovalevsky (1989). These researchers introduced the methods that can be used to define the properties of digital images, which became foundational in the development of this theory. Presently, digital topology plays a crucial role in fields where accurate topological analysis of digital structures is essential. These fields include artificial intelligence and pattern recognition.

There are two main approaches to the study of digital topology - the Graph-theoretic approach and the Axiomatic approach (Rosenfeld,1979). The former defines digital images as graphs while the latter defines digital images using mathematical axioms that can also be considered to be a first principle approach to the theory (Rosenfeld,1979).

2.8.1 A Digital Space

A *digital* $n-space$ $\mathbb{Z}^n$ is the $n-$tuple $(x_1, x_2, ..., x_n)$ of the Euclidean $n-$space having integer coordinates (Rosenfeld,1979). A point with integer coordinates is called a digital point. In computer grapics, the most commonly used representations are the 2- or 3-space, $\mathbb{Z}^2$ and $\mathbb{Z}^3$ respectively.

A *digital* $n-space$ $\mathbb{Z}^n$ consists of isolated, distinct points arranged in an integer lattice, rather than forming a continuous structure like in classical Euclidean spaces (Rosenfeld,1979). This discrete nature has significant implications for topology and spatial analysis in digital environments.

2.9 Graph-theoretic Digital Topology

The graph-theoretic approach to digital topology models digital spaces as graphs, where individual pixels (in 2D) or voxels (in 3D) are treated as nodes, and their adjacency relationships define connectivity and topological properties (Rosenfeld,1979). The following are some key definitions and properties that ensure its validity and applicability in digital topology.

2.9.1 Adjacency Relation

To study *nD* digital images, we say that two distinct points $p, q \in \mathbb{Z}^n$ are $k-\left(\text{or } k(t,n)-\right)$adjacent if for $t \in \mathbb{N}$ *s.t* $1 \leq t \leq n$ at most t of their coordinates differ by ± 1 and all the others coincide (Han, 2005).

We can obtain the $k-$adjacencies of $\mathbb{Z}^n$ as follows

$$k = \sum_{i=n-t}^{n-1} 2^{n-i} c_i^{\,n}, \text{ where } c_i^{\,n} = \frac{n!}{(n-i)!i!}$$

The configuration of the digital $k-$connectivity of $\mathbb{Z}^n$, $n = \{1, 2, 3\}$ are represented in Figure 2.3.

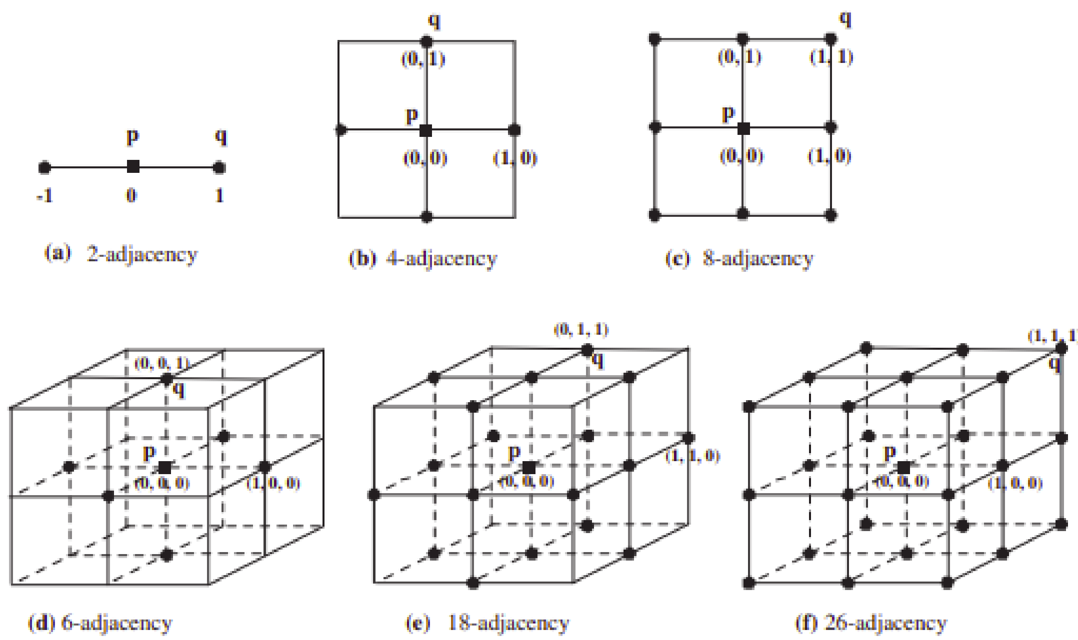

Figure 2.4: Configuration of the digital $k-$connectivity ***of*** $\mathbb{Z}^n$***, n ∈{1, 2, 3}***

The set $X \subset \mathbb{Z}^n$ with $k-$adjacency is called a digital image, denoted by (X,k). (Rosenfeld, 1979)

According to Rosenfeld (1979) the graph-based approach, and any approach for that matter, should be in agreement with classical topology, most especially with respect to connectedness and validity of the Jordan Curve Theorem. However, Slapal (2013), noted that neither the 4-adjacency nor the 8-adjacency as introduced by Rosenfeld, allows an analogue of the Jordan Curve Theorem (JCT). It is important that the JCT be definable in the digital plane since it guarantees a mathematical interpretation of the boundary properties of a space.

2.9.2 Digital Neighborhood

A digital $k-$neighborhood of a point $p \in \mathbb{Z}^n$ is the set $N_k(p) = \{q : p \text{ is } k-adjacent \text{ to } q\}$. Both the $k-$adjacency relation of $\mathbb{Z}^n$ (Han, 2005).

2.9.3 Digital Interval

For $a,b \in \mathbb{Z}^n$ with $a < b$, the set $[a,b] = \{n \in \mathbb{Z} : a \leq n \leq b\}$ with $2-$adjacencies is called a digital interval (Han, 2016).

Two subsets $(A,k),(B,k)\subset(X,k)$ are $k-$adjacent to each other if $A\cap B=\varnothing$ and there exists $a\in A,b\in B$ such that a,b are $k-adjacent$ to each other. $X\subset\mathbb{Z}^n$ is $k-$connected if there exists $A,B\in X$ such that $A\cup B=X$ *and* $A\cap B=\varnothing$

For a digital image (X,k), the $k-component$ of $x\in X$ is the largest $k-connected$ subset of (X,k) containing x (Han, 2016).

2.9.4 Path and Connectedness

The following definitions have been extracted from

A simple $k-path$ with $l+1$ elements in $\mathbb{Z}^n$ is an injective sequence $(x_i)_{i\in[0,l]_{\mathbb{Z}}}\subset\mathbb{Z}^n$ such that x_i and x_j are $k-adjacent$ *iff* $|i-j|=1$. If $x_0=x$ and $x_i=y$ then the length of the simple $k-path$, denoted by $l_k(x,y)$ is l (Ege & Karaca, 2015).

A $k-$path is called a $k-$arc if it has the additional property that for any two points p_i and p_j which are not end points $p_i\in N(p_j)$ implies that $|i-j|\leq 1$, that is, and arc is a path that does not intersect or touch itself with the possible exception of its end points (Eckhardt & Latecki, 2008).

$k-connected$ is an equivalence relation and hence this relation partitions X into equivalence classes, which are maximal.

Suppose X and Y are disjoint subsets of $\mathbb{Z}^n$. We say that X *surrounds* Y if any path from Y to the border of $\mathbb{Z}^n$ must meet X, where the border points of $\mathbb{Z}^n$ are elements of the complement of X.

Lemma 2.1

Let P be a path with two end points. Then there exists an arc P_o which is completely contained in P and has the same end points (Eckhardt & Latecki, 2003).

2.9.5 Grayscale Images

In digital topology, a grayscale image can be represented as a digital image where pixel intensities are defined on a discrete grid, and adjacency relations determine connectivity (Rosenfeld,1979). According to Kong & Rosenfeld (1989), a grayscale image is a function $I : X \to [0, L]$ where

i) $X \subseteq \mathbb{Z}^n$ is a finite subset of the integer lattice representing the image domain

ii) L is the maximum intensity level

iii) $I(x)$ represents the grayscale intensity value at pixel x, where 0 corresponds to black, L corresponds to white, and intermediate values represent shades of gray (Rosenfeld, 1979).

2.9.6 Adjacency Relations in Grayscale Images

As discussed in Section 2.7.1, adjacency relations between pixels determine connectivity on the image based on spatial relationships (Rosenfeld,1979). The adjacency relations on grayscale images will be as has been discussed for general picture subsets, i.e. the $k-$adjacencies of $\mathbb{Z}^n$ are defined by

$k = \sum_{i=n-t}^{n-1} 2^{n-i} c_i^n$, where $c_i^n = \frac{n!}{(n-i)!i!}$ which yields 4-adjacency, 8-adjacency, or m-adjacency. This type of adjacency is called spatial adjacency (Rosenfeld,1979).

However, in a grayscale image, these adjacency relations alone are insufficient, since neighboring pixels may have similar spatial connectivity but vastly different intensity values (Kong & Rosenfeld, 1989). Intensity-based adjacency takes into account the gray-level (intensity) similarity between pixels. Two pixels $p(x, y)$ and $q(x', y')$ are intensity-adjacent if their intensity values $I(p)$ and $I(q)$ satisfy

$$|I(p) - I(q)| \leq T \text{ where } T \text{ is a predefined threshold.}$$

This means that two pixels are considered adjacent only if their intensity values are close enough, regardless of whether they are spatially connected (Kong & Rosenfeld, 1989).

2.9.7 Graph-based Representation of a Grayscale Image

In Graph-based approach, a grayscale image can be modeled as an undirected graph $G=(V,E)$ where the vertices V represent pixels in the image, E define the adjacency relations (Kong & Rosenfeld, 1989).

Each vertex will possess a grayscale intensity value $I(v)\in[0,255]$ with 0 corresponding to a low intensity pixel (black) and 255 corresponding to the highest intensity pixel (white). All intermediate values represent shades of gray (Rosenfeld, 1979).

Example 2.7.1

A 3×3 grayscale image

$$\begin{array}{ccc} (50) & (120) & (200) \\ (80) & (150) & (220) \\ (90) & (180) & (250) \end{array}$$

will have a graphical representation with 4-adjacency relation as shown below

```
(50) --- (120) --- (200)
  |        |         |
(80) --- (150) --- (220)
  |        |         |
(90) --- (180) --- (250)
```

For the same image, the graphical representation with 8-adjacency relation will be

```
(50) --- (120) --- (200)
  |\       | /     / |
(80) --- (150) --- (220)
  | /      |  \    \ |
(90) --- (180) --- (250)
```

Appendix E of this thesis discusses the axiomatic approach to digital topology as developed by several authors including (Khalimsky, 1971; Kovalevsky, 1989; Rosenfeld, 1979).

2.10 Fuzzy Digital Topology

According to Rosenfeld (1979), the conventional approach of studying digital pictures, which involves the use segmentation processes to extract properties and relationships of a picture, represents a strong commitment when carried out for classical/ordinary sets. In many cases, it would be preferable to weaken this commitment by extracting fuzzy subsets rather than the ordinary subsets from the picture. Several topological concepts, including connectedness and surroundness of digital picture geometry have been extended by among other researchers Rosenfeld (1979).

2.10.1 Fuzzy Connectedness

Let A be a fuzzy subset of X (a rectangular array of integer coordinate points endowed with an adjacency relation), and let $\rho : x = x_0, x_1, ..., x_n = y$ be a path between two points x and y in X. Then

a) $S_A(\rho) = \min_{0 \le i \le n} A(x_i)$ is called the strength of path ρ

b) $C_A(x, y) = \max S_A(\rho)$ is called the degree of connectedness of x and y

Therefore,

a) $0 \le S_A(\rho) \le 1$ for all ρ

b) $0 \le C_A(x, y) \le 1$ for all x, y (2.39)

Fuzzy connectedness is not an equivalence relation

Proposition 2.10.1

C_A is reflexive and symmetric but not necessarily transitive

Proof

Reflexivity: For all $x \in X$ it is evident that $C_A(x, x) = A(x) = \min(A(x), A(x))$

Symmetry: It is clear since both C_A and $\min(A(x), A(y))$ are symmetric

Transitivity: Let X be the 1-by-3 array x, y, z and let $A(x) = A(z) = 1$, $A(y) < 1$ implying (x, y) and (y, z) are connected but x and z are not

Nevertheless, while fuzzy connectedness is not an equivalence relation, it is still a very useful relation on X since the notion of connected component with respect to the fuzzy set A may still be defined Rosenfeld (1979). This definition has many properties in common with the standard one even though the components do not constitute a partition.

2.10.2 Connected Components

In graph theory, connected components are sets of vertices within a graph where each vertex is reachable from every other vertex in the same component by traversing edges. In classical graph theory, the concept of connectivity is binary – either two vertices are connected by an edge, or they are not.

Fuzzy graph theory extends the concept of classical graph theory by allowing degrees of membership for vertices and edges, rather than just binary presence or absence. Fuzzy graphs are a way to model uncertainty or partial information in graph-based problems.

In a fuzzy graph, the concept of connected components can be adapted in different ways. Some of the meaningful ways that may be useful in image analysis, would be to consider degrees of connectivity and membership of pixels in a digital image. Consequently, instead of having components that are absolute, we can then have vertices and edges that are partially connected to one another based on their membership values.

2.10.3 Plateaus, Tops and Bottoms

A plateau in a fuzzy set A is a maximal connected subset of X on which A has constant value. i.e $A \subseteq X$ is a plateau if

a) A is connected

b) $A(x) = A(y)$ for all $x, y \in A$ (2.40)

c) $A(x) \neq A(y)$ for all pairs of neighbouring points $x \in A, y \notin A$

i) Clearly any $x \in X$ belongs to exactly one plateau.

ii) A plateau is called a *Top* if its value A is a local maximum, i.e $A(x) < A(y)$ for all pairs of neighbouring points $x \in A, y \notin A$.

iii) A plateau is called a *Bottom* if its value is a local minimum $A(x) > A(y)$ for all pairs of neighbouring points $x \in A, y \notin A$.

2.11 Surroundness

In digital topology, surroundness (also surroundedness) refers to the arrangement and connectivity of pixels or voxels in a digital image. The concept is essential in various image processing and computer vision tasks, such as segmentation, object recognition, and shape analysis. For example, in a 2-D digital image, a pixel can have 4 or 8 surrounding neighbors depending on whether diagonal connections are considered. In a 3D volume, similar considerations apply (Eckhardt & Latecki, 2003).

Definition 2.11.1 (Surround)

Suppose E is a rectangular array of integer-coordinate points endowed with an adjacency relation. If S and T are disjoint subsets of E, then S is said to surround T if any path from T to the border of E must meet S, (Rosenfeld, 1979)

It is clear from the definition that ‘surrounds’ is a strict order relation, i.e.,

a) Irreflexive: if S is nonempty, then S does not surround itself
b) Antisymmetric: if S surrounds T, the T does not surround S
c) Transitive: if S surrounds T and T surrounds W, the S surrounds W.

2.12 Topology and Fuzzy Logic in Image Analysis

This section discusses the significance of topology in digital imaging. It also summarises the application of fuzzy logic when dealing with uncertainty, and provides an overview of recent advancements in models that integrate graph-based representations and fuzzy set theory. The section also briefly discusses some fields of application and shows how fuzzy topology improves accuracy in image processing techniques.

2.12.1 Introduction to Digital Topology in Image Processing

In image processing, topology provides a mathematical foundation on which relationships such as connectivity, boundaries, and adjacency between pixels can be defined. As previously discussed digital topology deals with discrete structures, instead of continuous ones which therefore makes it very useful for image analysis (Kovalevsky, 1989).

Graph-based approaches play a crucial role in digital topology by representing images as networks in which pixels as vertices, and the adjacency relations between these pixels define connectivity (Rosenfeld, 1979). In image segmentation, this approach is considered to be more useful because it yields very accurate results. For example, when used in medical imaging, then the graph-based approaches can help to structure outputs in such a way that it becomes easier to detect anatomical boundaries as well as analyse cellular structures of images (Benhamou, 2024).

2.12.2 Fuzzy Topology in Image Analysis

If we allow pixels to have partial membership, then it is possible to move from classical digital topology that has strict boundaries to one that allows for gradual transitions between connected regions (Zadeh, 1965). When this is done, then an approach the reduces the uncertainties in digital pictures is created. According to Liu *et al*. (2024), key sources of uncertainty in image processing include

i) Grayness Ambiguity

This type of uncertainty arises from the difficulty in assigning a precise intensity value to a pixel. In medical imaging, such as X-rays, some variations in pixel intensity has been observed to accuracy of diagnosis. The dual characteristic of bipolar fuzzy sets is particularly effective in addressing this uncertainty. It enables a more accurate classification of pixel intensities and hence reduces uncertainty in grayscale images (Tizhoosh, 2005).

ii) Geometric Fuzziness

This type of uncertainty occurs when the boundaries of objects within an image are unclear. Consequently, it makies it difficult to establish the precise edges or shapes of the image. (Ilić, 2019).

iii) Incomplete Feature Knowledge

In some cases, image components share overlapping or missing features and will therefore be fuzzy. Consequently, it becomes difficult to carry out complete classification of features within a digital image (Chaira & Ray, 2017). In these cases, objects that share similarities may be classified differently

2.12.3 Fuzzy Image Processing Framework

In digital images, some pixels can sometimes be ambiguous and their membership in the digital picture cannot be determined with total certainty (Khan, 2017). Fuzzy image processing developed by many researchers including Khan (2017) is a computational approach that integrates fuzzy logic principles to deal with pixels that are considered to be ambiguous in digital images. Unlike conventional image processing methods that use crisp pixel intensities, fuzzy techniques enable an approach that considers the uncertainties in images and this can be seen in how the image data is interpreted. According to Khan (2017) the fuzzy image processing framework consists of three primary stages, each contributing to the refinement of image representation and analysis.

Stage 1: Fuzzification

The first step is called fuzzification. In this step pixel intensities are transformed into a fuzzy domain using membership functions (Zimmermaan, 2001). Instead of assigning to each pixel a fixed intensity value, this stage maps pixels to fuzzy sets (Khan, 2017). However, not all membership functions are appropriate in this transformation. The function used will depend on the complexity of the image, but also on the kind of output that is desired. Some of the commonly used membership functions include triangular, Gaussian, and sigmoid functions, because they help to define the boundaries between different intensity levels of pixels (Khan, 2017). This process will then model the gradual intensity transitions, and then reduce the ambiguity present in the image.

Stage 2: Modification of Membership Values

Once the image is in its fuzzy representation, the next stage involves the modification of membership values to achieve specific processing objective. Transformations, according to Tizhoosh (2005) are applied based on fuzzy rules. If contrast enhancement is needed, there are methods such as fuzzy histogram equalisation methods that may be used (Tizhoosh, 2005) . These will adjust membership functions and enhance visibility in dark spots of the images. For segmentation, fuzzy C-means (FCM) clustering is commonly used to classify pixels into different regions based on similarity in their fuzzy membership values (Szmidt & Kacprzyk, 2004).

Stage 3: Defuzzification (if necessary)

In many applications, the final processed fuzzy image must be converted back into the original representation. According to Khan (2017) this is done so that the image can be interpretated in a way that will be similar to the original one. To do this, a process referred to as defuzzification is applied. It was also noted by Szmidt and Kacprzyk (2004), that this will typically involve converting fuzzy membership values into the original intensity levels by using various techniques such as such as selecting the highest membership value (Khan, 2017). It can also involve centroid averaging, or applying another method called the mean-max method (Khan, 2017). This step is useful in medical imaging where the processed image must be interpreted by a person (Pal & King, 1983). However, Pal and King (1983) further notes that some fuzzy-based techniques may retain fuzzy values and there not fully defuzzify the image

2.13 Min-Max Normalisation in Image Processing

Min-max normalization is a widely used technique in image processing (Bezdek, 1981). It involves transforming data into a predefined range, usually $[0,1]$ or $[-1,1]$. This method, according to Szmidt and Kacprzyk (2004), is particularly effective whenever pixel intensity values need to be transformed in order to make the consistent across different datasets. The transformation ensures that all pixel values maintain relative intensity differences but also eliminate any variations that could affect other processing steps (Bezdek, 1981).

This sub-section reviews min-max normalisation in the context of fuzzy set theory, and highlights its applications.

2.13.1 Mathematical Formulation

As described by Szmidt and Kacprzyk (2004) the min-max normalisation ensures that data is linearly scaled within [0,1]. This is done by using a transformation that was defined by (Bezdek, 1981). This transformation is given by

$$x' = \frac{x - \min(x)}{\max(x) - \min(x)} \tag{2.41}$$

where;

i) x is the original data value,

ii) x' is the normalized value in the range [0,1] and

iii) $\min(x)$ and $\max(x)$ are the minimum and maximum values in the dataset

2.13.2 Applications in Fuzzy-Based Image Processing

Min-max normalisation is widely applied in fuzzy-based image processing. In this context, Szmidt and Kacprzyk (2004) notes that the process ensures that data is transformed into a standardised range that can be used in logic operations. By normalising image intensities then the effectiveness of fuzzy techniques is greatly enhanced.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Introduction

According to Rosenfeld (1979) the analysis of digital images often involves segmentation. In this process an image is divided into components that are considered relevant. It is then followed by measuring various properties and relationships among these segments. This process requires several key steps (Rosenfeld, 1979). They include;

i) Identifying the components that make up the image (Edward & Lang, 2000; Rosenfeld, 1979).
ii) Establishing adjacency relationships between these components, and
iii) Applying techniques that can extract the shapes that are required but while preserving topological properties such as connectedness (Rosenfeld, 1979).

These operations are essential for applications in image processing and pattern recognition.

While standard algorithms exist for tasks such as object counting, boundary detection, and thinning, Rosenfeld (1979) notes that their accuracy depend on the underlying topological framework. This framework is necessary because it governs the behavior of digital picture subsets (Edward & Lang, 2000). Establishing these fundamental topological properties is crucial because they ensure the process is reliable and produces accurate result in digital image analysis.

This chapter outlines the methodology used in the study. The sections present the approach that was taken in the course of doing this research. It will also provide the mathematical framework for digital fuzzy metric spaces that was used to develop the results in bipolar fuzzy context.

3.2 Essential Considerations

This research used a structured approach that was informed by key principles discussed in the subsequent subsections. These principles were firstly guided by the following considerations noted in (Kong & Rosenfeld, 1989; Rosenfeld, 1979);

3.2.1 Application-Oriented Theoretical Development

Any theory formulated for analysing the digital plane must be applicable across various real-world domains, including image processing (Kong & Rosenfeld, 1989; Rosenfeld, 1979). Therefore, we ensured that new axioms and definitions introduced in this study would contribute to a theoretical framework that can accurately model concepts of digital topology (Kong & Rosenfeld, 1989). The study prioritises the development of well-defined metrics that align with digital structures. We also attempt to maintain consistency with topological principles.

3.2.2 Consistency with Classical Topology

As described above, the methodology should ensure that the proposed approach remains consistent with classical topological principles (Kong & Rosenfeld, 1989; Rosenfeld, 1979). In particular approach must satisfy the conditions in relation to connectedness and the Jordan Curve Theorem (Khalimsky, 1971; Kopperman, 1995). Since connectedness plays a central role in image segmentation, the study extends bipolar fuzzy connectedness and provides results that align with traditional interpretations. Furthermore, following Rosenfeld (1979), the theory must preserve fundamental properties of topological spaces, while making sure that any deviation from classical results is well-justified and contributes to improving digital topology models.

3.2.3 Addressing Existing Limitations

Many existing digital topology models rely on ersatz topologies, such as Khalimsky topology, (Khalimsky, 1971; Kopperman, 1995)**.** This study aims to overcome these limitations by proposing enhanced digital fuzzy metric spaces that refine adjacency relations and incorporate bipolar fuzzy logic for better uncertainty representation. The methodology emphasises the need to improve existing topological frameworks as well as preserve their useful aspects in discrete digital environments.

By adhering to these methodological principles, this study aims to establish a coherent and applicable framework that enhances the mathematical foundation of digital topology and bipolar fuzzy metric spaces.

3.3 Fuzzy Techniques in Image Processing

Fuzzy set theory and fuzzy logic provide a mathematical model for approximate reasoning. Research in this area has proceeded along three main directions;

i) Fuzzy mathematical morphology in order to have a theoretical point of view in image processing
ii) Research around filters for noise reduction on in images
iii) In similarity measures for image comparison

Similarity measures play a very central role in image retrieval applications. The definition of an adequate similarity measure for measuring the similarity between fuzzy sets is of great importance in the fields of image processing and pattern recognition (Omran & Hassaballah, 2007). An important problem in image processing is the comparison of images. According to Van der Weken et.al., (2004), if different algorithms are applied to an image, then an objective measure will be needed to make accurate comparison with the output images.

Since gray-scale images can be identified with fuzzy sets, similarity measures may be used to retrieve the images from the database that are most similar, in certain predetermined criteria, to this source image. There is no generic method for selecting a suitable similarity or distance measure. However, prior knowledge of relevant features may be used to make this selection or establish a new measure.

Therefore, the definition of different kinds of distance measures will be very essential in this research. Such measures guarantee the utility of proposed theory in application problems, especially in image analysis and processing and artificial intelligence.

3.4 Computational Implementation using Python

To validate the theoretical findings, this study implements digital bipolar fuzzy metric spaces and image processing algorithms using Python.

3.4.1 Python Libraries

Python libraries are collections of pre-written code that provide functionalities for various tasks, such as data manipulation, machine learning, image processing, and mathematical computations.

These libraries save time and effort by allowing users to use existing functions instead of writing everything from scratch (Python Software Foundation, 2023).

Several Python libraries will be essential in this study. They are;

i) **NumPy (Numerical Python)**: a powerful library for multi-dimensional array manipulation and numerical computing. It is particularly useful for implementing bipolar fuzzy distance measures in image analysis, as it provides:
   - Fast array operations (vectorized calculations) instead of slow Python loops.
   - Broadcasting capabilities for applying fuzzy transformations across entire images.
   - Efficient memory management for large datasets and images.

ii) **OpenCV (Open-Source Computer Vision Library)**: a library for computer vision and image processing tasks. It is essential in this study for:
   - Loading and processing images from various formats.
   - Applying noise reduction filters to improve fuzzy set calculations.
   - Extracting features (edges, contours, and connected components) to evaluate digital fuzzy metric spaces.

iii) **SciPy (Scientific Python)**: built on NumPy and provides advanced scientific computing capabilities, particularly for image processing, statistical analysis, and mathematical optimizations. It is crucial in this study for:
   - Computing entropy-based image segmentation for digital fuzzy metric spaces.
   - Applying graph-theoretic approaches to model adjacency relationships in digital images.

iv) **Matplotlib**: useful in data visualization and helps to analyze and display fuzzy topology results graphically. It is used for:
   - Visualizing fuzzy segmentations and connected components.
   - Plotting bipolar fuzzy membership distributions in digital topology analysis.
   - Comparing original and processed images to evaluate segmentation quality

CHAPTER FOUR
RESULTS AND DISCUSSISONS

4.1 Introduction

There are various approaches to defining a fuzzy metrics. These approaches differ based on context and application, as discussed in Chapter Two. In most cases, these fuzzy metrics serve as natural extensions of their non-fuzzy concepts. They include the fuzzy Euclidean distance, the fuzzy Jaccard similarity, the fuzzy cosine similarity and the fuzzy Hausdorff distance among others.
Distance and similarity measures are widely used in pattern recognition, machine learning, image processing and other related fields. As mentioned in Chapter Two, the outcome of a distance function quantifies the difference between the values within fuzzy sets. Intuitively, a distance of zero indicates that the fuzzy sets are identical, while greater distances reflect increasing dissimilarity between the fuzzy sets under discussion.

The properties of a metric are critical because they strictly define the distance between points in the given metric space. In that regard, a distance measure is only a metric if it satisfies reflexivity, separability, symmetry, and the triangle inequality (Charlotte, 2016). Therefore, while every metric is a function of distance, it is well known that not all distance measures meet the criteria to be considered metrics.

This chapter begins by introducing the concept of measures of bipolar fuzziness and explores their fundamental properties. Similarly to Charlotte (2016) it examines how these measures extend traditional fuzzy set theories and assesses their mathematical consistency and applicability. Additionally, the chapter evaluates the coherence of the results, with the aim of making sure that they align with established fuzzy logic principles and that they contribute to a more comprehensive understanding of bipolar fuzziness in various contexts.

The chapter concludes by showing the application of the derived results through the introduction of a bipolar fuzzy-based computational approach. This approach extends classical fuzzy segmentation models to bipolar fuzzy segmentation and makes use of the bipolar fuzzy Jaccard similarity measure, as applied to a pair of grayscale digital images.

4.2 Measures of Bipolar Fuzzy Sets

Most real-life phenomena are subjective. The need to measure and compare similarities and differences in real life situation has made similarity measures in fuzzy set theory to be very crucial (Charlotte, 2016). These measures provide a mathematical framework for modeling uncertain information and also for making them essential in various fields such as decision-making, pattern recognition, machine learning, and artificial intelligence.

Two types of measures of fuzziness have been introduced on fuzzy sets: similarity measures and distance measures (Kaufmann, 1975; Szmidt & Kacprzyk, 2000; Yager, 1979). The differences between these have been discussed in Section 2.4. In the following subsections, the extensions of these notions will be proposed and discussed. The aim is to develop a framework within which comparisons of bipolar fuzzy sets may be done and also within which metric spaces may be defined.

The measure of bipolar fuzziness in a bipolar fuzzy set reflects its degree of bipolarity. Essentially, this would be a mapping that satisfies certain conditions consistent with those proposed by Yager (1979). Additionally, various measures of fuzziness have been studied (Kaufmann, 1975; Szmidt & Kacprzyk, 2000; Yager, 1979). The extension provided here for bipolar fuzzy sets builds on foundations from fuzzy and intuitionistic fuzzy sets.

4.2.1 Similarity Measure of Bipolar Fuzzy Sets

Similarity measures are complex and context-dependent Charlotte (2016). As such, several different methods have been developed. The utility of these methods is found in several areas of application including pattern recognition and clustering.

Fundamentally, a similarity function in the context of bipolar fuzzy sets, is a function that can evaluate how closely two bipolar fuzzy sets share the same positive and negative membership values across the universe of discourse. This idea can be formulated as follows.

Definition 4.2.1

Let A, B be two bipolar fuzzy subsets of X. A similarity measure on X is a function $S : A \times B \rightarrow [0,1]$ such that;

a) $S(A,B)=1$ whenever $A^{+}=B^{+}$ and $A^{-}=B^{-}$ $\forall x\in X$ (4.1)

b) $S(A,B)=S(B,A)$ (4.2)

c) If $A,B\neq\emptyset$, then $S\left(A^{+},B^{+}\right)>0$ and $\mathrm{S}\left(A^{-},B^{-}\right)<0$, otherwise $S(A,B)=0$ (4.3)

The similarity measure for bipolar fuzzy sets is inherently a relative concept, meaning its interpretation depends on the specific context in which it is applied rather than adhering to a fixed, absolute standard. Unlike traditional metrics that provide a universal measure of similarity, bipolar fuzzy similarity measures are influenced by factors. These include the nature of the fuzzy sets being compared, the criteria used for evaluation, and the specific application domain.

Given how relative similarity measures in bipolar fuzzy sets are, it then becomes essential to adopt what has already been established so as to align with their unique characteristics. One widely used similarity measure that aligns with these principles is the Jaccard similarity, which we now characterize in the context of bipolar fuzzy sets.

4.2.2 Characterisation of the Jaccard Similarity on Bipolar Fuzzy Sets

The Jaccard similarity measure is widely used in fuzzy set theory for quantifying the degree of overlap between sets while considering differences in their membership values. In the context of bipolar fuzzy sets, traditional Jaccard similarity must be adapted to account for both positive and negative membership degrees. This will enable the formulated result to provide a more accurate and similarity assessment**.**

Definition 4.2.2

Let A,B be two bipolar fuzzy sets. Using the set operations in Section 2.5.1, the Jaccard similarity on A and B may be defined as

$$S(A,B)=\sum_{i=1}^{n}\left(\frac{A^{+}(x_i)\wedge B^{+}(x_i),A^{-}(x_i)\wedge B^{-}(x_i)}{A^{+}(x_i)\vee B^{+}(x_i),A^{-}(x_i)\vee B^{-}(x_i)}\right) \tag{4.4}$$

This characterisation satisfies the three properties of a similarity measure as listed in Definition 4.2.1 above, that is;

a) Suppose $A^{+}=B^{+}$ and $A^{-}=B^{-}$ $\forall x\in X$. Then

$$S(A,B)=\sum_{i=1}^{n}\left(\frac{A^{+}(x_i),A^{-}(x_i)}{A^{+}(x_i),A^{-}(x_i)}\right)=1 \tag{4.5}$$

b)

$$\begin{aligned}S(A,B)&=\sum_{i=1}^{n}\left(\frac{A^{+}(x_i)\wedge B^{+}(x_i),A^{-}(x_i)\wedge B^{-}(x_i)}{A^{+}(x_i)\vee B^{+}(x_i),A^{-}(x_i)\vee B^{-}(x_i)}\right)\\&=\sum_{i=1}^{n}\left(\frac{B^{+}(x_i)\wedge A^{+}(x_i),B^{-}(x_i)\wedge A^{-}(x_i)}{B^{+}(x_i)\vee A^{+}(x_i),B^{-}(x_i)\vee A^{-}(x_i)}\right)\\&=S(B,A)\end{aligned} \tag{4.6}$$

c) Suppose $A,B\neq\varnothing$. Then $A^{+}(x_i)\wedge B^{+}(x_i)\neq 0$ and $A^{-}(x_i)\wedge B^{-}(x_i)\neq 0$. Hence

$$S(A^{+},B^{+})>0 \text{ and } S(A^{-},B^{-})<0 \tag{4.7}$$

In set theory, the Jaccard similarity measures the degree of overlap between two sets by calculating the ratio of their intersection to their union (Kaufmann, 1975; Szmidt & Kacprzyk, 2000; Yager, 1979). This concept has been extended to bipolar fuzzy sets, where a similar approach is used to assess the relationship between fuzzy elements. Similarity measures serve as a foundation for developing entropy measures. The next subsection introduces the bipolar fuzzy entropy measure, which extends traditional entropy formulations.

4.3 Bipolar Fuzzy Entropy

Fuzzy entropy is a means of determining the amount of information present in a fuzzy set. In de Luca and Termini (1972) a version of the entropy of a fuzzy set is given. This section explores the properties that define a suitable entropy measure for bipolar fuzzy sets.

Proposition 4.3.1

A valid entropy measure for bipolar fuzzy sets should satisfy the following key properties:

i) Minimum entropy for crisp sets

$$d(A)=0 \text{ if } A \text{ is crisp.} \tag{4.8}$$

If a bipolar fuzzy set is crisp (i.e., membership values are strictly 0 or 1), then its entropy should be zero, indicating no uncertainty.

ii) Maximum entropy at maximum uncertainty

Entropy reaches its maximum when the membership values are evenly distributed (i.e., maximum fuzziness)

$d\left(A^{+}\right)$ takes a maximum value whenever $\mu^{+}(x)=0.5$ and (4.9)

$d\left(A^{-}\right)$ takes a minimum value whenever $\mu^{-}(x)=-0.5$

iii) Symmetry

The entropy of a bipolar fuzzy set should remain unchanged if the membership functions are swapped symmetrically, i.e.,

$d(A) \geq d(A')$ if $\mu_{A}{}^{+}(x) \leq \mu_{A'}{}^{+}(x)$ for $\mu_{A}{}^{+}(x)=0.5$ and

$d(A) \leq d(A')$ if $\mu_{A}{}^{-}(x) \geq \mu_{A'}{}^{-}(x)$ for $\mu_{A}{}^{-}(x)=-0.5$ (4.10)

iv) Complementary entropy relation

The entropy of a bipolar fuzzy set should relate to the entropy of its complement A^{c} ensuring consistency in uncertainty measurement:

$$d\left(A^{c}\right)=d(A)=\begin{cases} d\left(A^{+}\right)=d\left(\left(A^{c}\right)^{+}\right) \text{ and} \\ d\left(A^{-}\right)=d\left(\left(A^{c}\right)^{-}\right) \end{cases} \quad \text{where } A^{c} \text{ is the complement of } A \tag{4.11}$$

These properties are an extension of those introduced by de Luca and Termini (1972) in the realm of fuzzy sets. Building upon these properties introduced we establish the following definition, which formalises the entropy measure for bipolar fuzzy sets.

Definition 4.3.1

Let A be a bipolar fuzzy subset of X . The entropy of A may be defined as

$$H(A)=-\sum_{x\in X} h_{A}^{+}(x)+h_{A}^{-}(x) \text{ where}$$

$$h_{A}^{+}(x)=\mu_{A}^{+}(x)\ln\mu_{A}^{+}(x)+\left(1-\mu_{A}^{+}(x)\right)\ln\left(1-\mu_{A}^{+}(x)\right)$$

$$\text{and} \tag{4.12}$$

$$h_{A}^{-}(x)=\mu_{A}^{-}(x)\ln\mu_{A}^{-}(x)+\left(1-\mu_{A}^{-}(x)\right)\ln\left(1-\mu_{A}^{-}(x)\right)$$

To confirm that the definition of bipolar fuzzy entropy as provided is a valid entropy measure, we must verify that it satisfies key entropy properties in Proposition 4.3.1; non-negativity, maximum entropy at maximum uncertainty, minimum entropy for crisp sets, symmetry. Monotonicity

This definition is consistent with the boundary behaviour discussed in the classical entropy measure, i.e., when $\mu_A^{+}(x)=1$ and $\mu_A^{0}(x)=0$, then

$$h_A^{+}(x)=1.\ln(1)+0.\ln(0)=0 \text{ and } h_A^{-}(x)=0.\ln(0)+1.\ln(1)=0$$

$$H(A)=-\sum_{x\in X}0+0=0 \tag{4.13}$$

Note: in entropy the $0.\ln(0)=0$

To confirm that the definition of bipolar fuzzy entropy as provided is a valid entropy measure, we must verify that it satisfies key entropy properties in Proposition 4.3.1; non-negativity, maximum entropy at maximum uncertainty, minimum entropy for crisp sets, symmetry and Monotonicity.

i) Non-negativity

Since $\mu_A^{+}(x), 1-\mu_A^{+}(x)\in[0,1]$ then $\ln\mu_A^{+}(x)\leq 0, \ln\left(1-\mu_A^{+}(x)\right)\leq 0$

$$h_A^{+}(x)=\mu_A^{+}(x)\ln\mu_A^{+}(x)+\left(1-\mu_A^{+}(x)\right)\ln\left(1-\mu_A^{+}(x)\right)<0$$

Similarly, since $\mu_A^{-}(x), 1-\mu_A^{-}(x)\in[-1,0]$ then $\ln\left|\mu_A^{-}(x)\right|\leq 0, \ln\left|\left(1-\mu_A^{-}(x)\right)\right|\leq 0$

$$h_A^{-}(x)=\mu_A^{-}(x)\left|\ln\mu_A^{-}(x)\right|+\left(1-\mu_A^{-}(x)\right)\ln\left|\left(1-\mu_A^{-}(x)\right)\right|\leq 0$$

Therefore $H(A)=-\sum_{x\in X}h_A^{+}(x)+h_A^{-}(x)\geq 0$ (4.14)

ii) Maximum entropy at maximum uncertainty

Intuitively, $H(A)=-\sum_{x\in X}h_A^{+}(x)+h_A^{-}(x)\geq 0$ is at maximum when membership values are at their most uncertain state. This will occur when membership values are equally spread between 0 and 1 or when an element is equally likely to belong to the set or not

In the bipolar fuzzy set this will happen when $h_A^+(x) = 0.5$ and $h_A^-(x) = -0.5$

iii) Minimum entropy for crisp sets

Suppose $\mu_A^+(x) = \mu_A^-(x) = 0$ then $h_A^+(x) = h_A^-(x) = 0$ and $H(A) = -\sum_{x\in X} h_A^+(x) + h_A^-(x) = 0$

Similarly if $\mu_A^+(x) = \mu_A^-(x) = 1$ then $\ln \mu_A^+(x) = 0$ hence $h_A^+(x) = h_A^-(x) = 0$ and

$H(A) = -\sum_{x\in X} h_A^+(x) + h_A^-(x) = 0$ implying that A is crisp (4.15)

iv) Symmetry

By definition,

$$\begin{aligned} \mu_{A^c}^+(x) &= 1 - \mu_A^+(x) \\ \mu_{A^c}^-(x) &= 1 - \mu_A^-(x) \end{aligned} \tag{4.16}$$

Therefore,

$$h_{A^c}^+(x) = \left(1 - \mu_A^+(x)\right)\ln\left(1 - \mu_A^+(x)\right) + \mu_A^+(x)\ln \mu_A^+(x) = h_A^+(x)$$

$$h_{A^c}^-(x) = \left(1 - \mu_A^-(x)\right)\ln\left(1 - \mu_A^-(x)\right) + \mu_A^-(x)\ln \mu_A^-(x) = h_A^-(x)$$

$$H(A) = -\sum_{x\in X} h_A^+(x) + h_A^-(x) = \sum_{x\in X} h_{A^c}^+(x) + h_{A^c}^-(x) = H\left(A^c\right) \tag{4.17}$$

Thus, entropy is invariant under complementation.

v) Monotonicity

Let A, B be a bipolar fuzzy subset of X and B has membership values closer to 0.5 i.e. $h_B^+(x)$ and $h_B^-(x)$ are closer to 0.5 than $h_A^+(x)$ and $h_A^-(x)$. Since entropy is maximised at 0.5 then $H(B) \geq H(A)$

The concept of bipolar fuzzy entropy extends the classical notion of fuzzy entropy, originally introduced by Zadeh (1965), to account for both positive and negative membership degrees within bipolar fuzzy sets. While traditional fuzzy entropy quantifies the uncertainty or fuzziness within a standard fuzzy set, bipolar fuzzy entropy seem to provide a more comprehensive framework by incorporating dual aspects of membership. Consequently, we obtain a much richer interpretation of relationships. This generalisation makes it possible for us to model applications that require interpretations of opposing or complementary attributes that are relative. They include areas such as sentiment analysis, medical diagnosis, and decision-making systems.

4.3.1 Summary of Entropy Properties Applied to Bipolar Fuzzy Sets

The table below summarises the entropy properties as defined on bipolar fuzzy sets.

Table 4.1: Entropy Properties Applied to Bipolar Fuzzy Sets

Entropy Property	Definition	Application to Bipolar Fuzzy Entropy
Non-Negativity	$H(A) \geq 0$	Entropy is not negative.
Maximum Entropy	Entropy is max when $h_A^+(x) = 0.5$ and $h_A^-(x) = 0.5$	Bipolar fuzzy entropy is maximized when uncertainty **is** maximum
Minimum Entropy (Crisp Sets)	$H(A) = 0$ if A is a crisp set	If $\mu_A^+(x), \mu_A^-(x)$ are 0 or 1, then entropy should be zero
Symmetry	$H(A) \geq H(A^c)$	Bipolar fuzzy entropy is invariant under complement**.**

4.4 Extending Rényi Entropy to Bipolar Fuzzy Sets

While traditional Shannon entropy quantifies uncertainty in a system, Rényi entropy, discussed in Chapter two, provides a more generalized and flexible entropy measure by introducing a tunable order parameter α which controls the sensitivity to different probability distributions. In this section, we formulate a bipolar fuzzy Rényi entropy. This formulation provides us with a significant advantages in image processing because it helps to deal with noise that is present in images. In particular it is useful for complex visual data interpretation.

Firstly, to extend Rényi entropy to bipolar fuzzy sets, we define a new entropy function that considers both the positive and negative membership degrees of elements in a BFS. This formulation begins by defining a bipolar fuzzy probability distribution, provides a characterization of the entropy and investigates its consistency with the general properties of entropy.

4.4.1 Bipolar Fuzzy Probability Distribution

Given a bipolar fuzzy set A, the bipolar fuzzy probability distribution may be defined by

$$p_i^+ = \frac{\mu^+(x_i)}{\sum_{j=1}^{n} \mu^+(x_j)}$$ where p_i^+ represents the positive probability distribution

$$p_i^- = \frac{|\mu^-(x_i)|}{\sum_{j=1}^{n} |\mu^-(x_j)|}$$ where p_i^- represents the negative probability contribution

Using these probability distributions, we can define the bipolar Rényi entropy (BRE) by

$$H_\alpha^B(A) = \frac{1}{1-\alpha} \ln\left(\sum_{i=1}^{n} (p_i^+)^\alpha + \sum_{i=1}^{n} (p_i^-)^\alpha \right) \tag{4.18}$$

This formulation ensures that the definition encapsulates both the supporting and opposing membership degrees.

4.4.2 Properties of the Proposed BRE

The proposed Bipolar Rényi Entropy (BRE) in equation 4.18 is an extension of the Rényi entropy that can now accommodate the characteristics of bipolar fuzzy sets. The following properties highlight key aspects of the proposed BRE. They demonstrate that the BRE is consistent with what already exists, i.e., the Rényi entropy measure

i) If $\mu^-(x_i)\forall i$ then $H_\alpha^B(A)$ is equal to the classical entropy. In other words the proposed BRE for bipolar fuzzy sets is a general form of the standard Rényi entropy (Kaufmann, 1975; Szmidt & Kacprzyk, 2000; Yager, 1979).

ii) The parameter α enables dynamic control over the entropy computation, with higher α emphasizing dominant elements (Yager, 1979).

To examine the proposed BRE further, we analyse its mathematical properties. This will also be a way of showing that the BRE is valid.

i) **Non-negativity**

Any entropy measure must be non-negative i.e., $H_\alpha^B(A) \geq 0$

Proof

$\forall i, 0 \leq p_i^+ \leq 1$ and $0 \leq p_i^- \leq 1$. Since $\alpha > 0$ then $0 \leq \left(p_i^+\right)^\alpha \leq 1$ and $0 \leq \left(p_i^-\right)^\alpha \leq 1 \ \forall i$

Therefore $$H_\alpha^B(A) = \frac{1}{1-\alpha} \ln\left(\sum_{i=1}^{n}\left(p_i^+\right)^\alpha + \sum_{i=1}^{n}\left(p_i^-\right)^\alpha\right) \geq 0 \tag{4.19}$$

ii) Monotonicity

BRE, like classical entropy measures, is expected to be monotonically decreasing in α meaning that as the probability contribution increases with increase in α, the entropy associated with it decreases. The interpretation of this behaviour is a reduction in uncertainty.

Proof

Consider the function

$$S_\alpha = \sum_{i=1}^{n}\left(p_i^+\right)^\alpha + \sum_{i=1}^{n}\left(p_i^-\right)^\alpha \tag{4.20}$$

Taking the derivative with respect to α

$$\frac{d}{d\alpha} S_\alpha = \sum_{i=1}^{n}\left(p_i^+\right)^\alpha \ln\left(p_i^+\right) + \sum_{i=1}^{n}\left(p_i^-\right)^\alpha \ln\left(p_i^-\right) \tag{4.21}$$

Since $0 \leq p_i^+ \leq 1$ and $0 \leq p_i^- \leq 1 \ \forall i$ then their logarithms will be negative

$$\frac{d}{d\alpha} S_\alpha < 0 \tag{4.22}$$

Which implies S_α is monotonically decreasing in α. Therefore,

$$H_\alpha^B(A) = \frac{1}{1-\alpha} \ln\left(\sum_{i=1}^{n}\left(p_i^+\right)^\alpha + \sum_{i=1}^{n}\left(p_i^-\right)^\alpha\right) \text{ is monotonically decreasing} \tag{4.23}$$

Regarding the monotonicity of entropy, it is important to note that maximum entropy occurs when the probability distribution is uniform, and the entropy decreases as the probability distribution becomes more skewed towards a few specific events. Therefore, intuitively, with more information about a system, the uncertainty about its state decreases, leading to a decrease in entropy.

iii) Concavity

Entropy is generally expected to be concave. To establish that BRE is concave, we need to compute its second derivative and show that it is non-positive i.e., $\frac{d^2}{d\alpha^2}\left(H_\alpha^B(A)\right) \leq 0$

Proof

Differentiating $S_\alpha = \sum_{i=1}^{n}\left(p_i^+\right)^\alpha + \sum_{i=1}^{n}\left(p_i^-\right)^\alpha$ a second time

$$\frac{d}{d\alpha}\left(\frac{d}{d\alpha}S_\alpha\right) = \frac{d}{d\alpha}\left(\sum_{i=1}^{n}\left(p_i^+\right)^\alpha \ln\left(p_i^+\right) + \sum_{i=1}^{n}\left(p_i^-\right)^\alpha \ln\left(p_i^-\right)\right)$$

$$= \sum_{i=1}^{n}\left(p_i^+\right)^\alpha\left(\ln p_i^+\right)^2 + \sum_{i=1}^{n}\left(p_i^-\right)^\alpha\left(\ln p_i^-\right)^2 - \frac{\sum_{i=1}^{n}\left(p_i^+\right)^\alpha\left(\ln p_i^+\right) + \sum_{i=1}^{n}\left(p_i^-\right)^\alpha\left(\ln p_i^-\right)^2}{\sum_{i=1}^{n}\left(p_i^+\right)^\alpha + \sum_{i=1}^{n}\left(p_i^-\right)^\alpha}$$

$$\leq 0, \text{ since } 0 \leq p_i^+ \leq 1 \text{ and } 0 \leq p_i^- \leq 1 \quad \forall i \tag{4.24}$$

4.5 Digital Fuzzy Metric Space

In digital spaces, where data is often discrete and pixel-based, adapting fuzzy metrics to digital structures can potentially medical imaging. Digital fuzzy metric spaces make clear the notion of distance by integrating fuzzy logic principles (Tversky, 1977). Consequently, more nuanced similarity and proximity measures that align with the inherent uncertainty in digital data may be formulated.

To formally establish the concept of a digital fuzzy metric space, we define its structural components in terms of fuzzy logic and adjacency relationships as shown by Tversky (1977) . The following definition outlines the fundamental properties that characterise this space.

Definition 4.4.1

The triple $\left(\mathbb{Z}^n, M, *\right)$ is called a digital fuzzy metric space if $*$ is a continuous t – norm, $(X,k) \subset \mathbb{Z}^n$ is a fuzzy subset of $\mathbb{Z}^n$ with k - adjacency such that $\forall x, y, z \in X$ and $t, s > 0$

i) $M(x,y,t) > 0$

ii) $M(x,y,t) = 1$ iff $x = y$

iii) $M(x,y,t) = M(y,x,t)$ (4.25)

iv) $M(x,z,t+s) \leq M(x,y,t) * M(y,z,s)$

v) $M(x,y,\cdot):(0,\infty) \to [0,1]$ is fuzzy continuous

where the k - adjacency of X is defined by $k = \sum_{i=n-t}^{n-1} 2^{n-i} c_i^n$ with $c_i^n = \dfrac{n!}{(n-i)!i!}$

This definition is a novel extension of the fuzzy metric defined by George and Veeramani (1994) with the extension being on a digital fuzzy set with a given adjacency relation. The extension specifically incorporates digital fuzzy sets, integrating a predefined adjacency relation to better capture spatial proximity and uncertainty in digital environments. By adapting the classical fuzzy metric framework to digital structures, this approach could enhance.

Equivalently, suppose $(X,k) \subset \mathbb{Z}^n$ is a fuzzy digital image of k - adjacency. Given the Euclidean metric d then the triple (X,d,k) will be a metric induced by the Euclidean metric in the fuzzy digital image.

Proposition 4.4.1

Suppose $(X,k) \subset \mathbb{Z}^n$ and (X,d,k) is a digital fuzzy metric space. Let x_i, x_j be points in a sequence $\{x_n\} \subset X$ such that they are k - adjacent i.e., $x_i \in N(x_j,1)$ or $x_j \in N(x_i,1)$ and $x_i \neq x_j$. Then they have the Euclidean distance $d(x_i,x_j) \geq 1$ or $d(x_i,x_j) \leq \sqrt{t}$, $1 \leq t \leq 2$.

Proof

The Euclidean distance between two points $x_i, x_j \in (X, d, k)$ is given by

$$d(x_i, x_j) = \sqrt{(x_i^1 - x_j^1)^2 + (x_i^2 - x_j^2)^2 + \ldots + (x_i^n - x_j^n)^2}$$

Since x_i, x_j are distinct and (X, d, k) is k - adjacent then the maximum distance along any axis is at most 1 i.e.,

$$d(x_i, x_j) \le \sqrt{1^2 + 1^2 + \ldots + 1^2} = \sqrt{t} \text{ for } 1 \le t \le 2 \tag{4.26}$$

4.6 Bipolar Fuzzy Metrics

In transitioning from digital fuzzy metric space to bipolar fuzzy metric space, it is essential to refine the representation of uncertainty by incorporating both positive and negative membership degrees. A few preliminary definitions are necessary in order to establish a formal mathematical framework for bipolar fuzzy metrics.

4.6.1 Path Strength

Path strength is a key concept in the formulation of bipolar fuzzy metrics because it provides a structured way to quantify connectivity and uncertainty in a bipolar fuzzy graph.

A path $\rho : x = x_0, x_1, \ldots, x_n = y$ from x to y in a bipolar fuzzy graph $(\mu, \tilde{R})$ is a sequence of distinct vertices $x_0, x_1, \ldots, x_n$ such that

$$\begin{aligned} &\tilde{R}^+(x_{i-1}, x_i) > 0 \\ &\tilde{R}^-(x_{i-1}, x_i) > -1 \quad \forall i = 1, 2, \ldots, n \end{aligned} \tag{4.27}$$

To define the strength of a path in the bipolar fuzzy subset, we have to consider both the strength of the path in the direction representing the satisfaction degree from x to y and the direction representing the satisfaction of x and y to the implicit counter-property. Hence, we have the following formulation.

Definition 4.6.1

Let μ be a bipolar fuzzy subset of X and $\rho = \{v_o, v_1, ..., v_n\}$ be any path between two points $u, v \in \mu$. The $\mu^+ -$strength of ρ denoted $S_{\mu^+}(\rho)$ is given by

$$S_{\mu^+}(\rho) = \min\left\{R^+(v_{i-1}, v_i) \middle| 1 \le i \le n\right\} \tag{4.28}$$

The $\mu^- -$strength of ρ denoted $S_{\mu^-}(\rho)$ is given by

$$S_{\mu^-}(\rho) = \max\left\{R^-(v_{i-1}, v_i) \middle| 1 \le i \le n\right\}. \tag{4.29}$$

Here $\mu^- -$strength is the weight of the path ρ with respect to the counter-property of $\mu^+ -$strength. Obviously, $S_{\mu^+}(\rho) \in [0,1]$ and $S_{\mu^-}(\rho) \in [-1, 0]$.

If there exists a path between u and v then the two points are said to be connected.

4.6.2 Length

The length $\ell(\rho)$ of a path from x to y is the number, $n \ge 0$ of vertices or nodes between the points. If the path has length 0, it is convenient to define its strength to be;

$$\begin{aligned} S_{\mu^+}(\rho) &= \mu^+(x_0) \\ S_{\mu^-}(\rho) &= \mu^-(x_0) \end{aligned} \tag{4.30}$$

4.6.3 A Metric on Bipolar Fuzzy Graphs

In the context of bipolar fuzzy graphs, the concepts of path strength and path length play a fundamental role in defining a meaningful notion of distance. Path strength quantifies the overall connectivity between nodes, whereas path length captures the structural properties of a sequence of connections. Together, these formulations provide a basis for defining a distance function that reflects the uncertainty and bipolar nature of the graph. We propose the following intuitive distance concept in bipolar fuzzy setting.

Definition 4.6.2 $\mu^+ -$distance (resp. $\mu^- -$distance)

The μ^+ – distance (resp. μ^- – distance), $dis_{\mu^+}(x,y)\left(\text{resp. } dis_{\mu^-}(x,y)\right)$ is the length of the shortest μ^+ – strongest (resp. μ^- – strongest) path between x and y

Since a well-defined distance function is crucial for establishing a bipolar fuzzy metric space, it is necessary to verify whether the proposed distance satisfies the axioms of a metric—namely, non-negativity, symmetry, and the triangle inequality. The following analysis explores these conditions to ensure that the defined distance functions are mathematically valid as metrics within the bipolar fuzzy framework.

Corollary 4.6.1

The proposed distance in definition 4.6.2 does not define a metric on Bipolar Fuzzy graphs.

Proof

a) Symmetry: $dis_{\mu^+}(x,y) = dis_{\mu^+}(y,x)\left(\text{resp. } dis_{\mu^-}(x,y) = dis_{\mu^-}(x,y)\right)$.

b) Nonnegative: $dis_{\mu^+}(x,y) \geq 0\left(\text{resp. } dis_{\mu^-}(x,y) \geq 0\right)$ $\forall x \neq y$.

   Clearly $dis_{\mu^+}(x,x) = 0\left(dis_{\mu^-}(x,x) = 0\right)$ since any path ρ from x to x has strength $S_{\mu^+}(\rho) = \mu^+(x)\left(\text{resp. } S_{\mu^-}(\rho) = \mu^-(x)\right)$ which is the strength of the path of length 0

c) This notion of distance on the bipolar fuzzy graph does not satisfy the triangle property since using a counterexample;

   Let $G = (V, E)$ be a bipolar fuzzy graph with $V = \{x, y, z\}$ such that

   $$\begin{matrix} \mu^+(x,y) = 0.6, \mu^+(y,z) = 0.6, \mu^+(x,z) = 0.2 \\ \mu^-(x,y) = -0.2, \mu^-(y,z) = -0.2, \mu^-(x,z) = -0.5 \end{matrix} \text{ then}$$

   $dis_{\mu^+}(x,z) = 1 - \mu^+(x,z) + \mu^-(x,z) = 1 - 0.2 - 0.5 = 0.3$ and

   $dis_{\mu^+}(x,y) + dis_{\mu^+}(y,z) = [1 - 0.6 - 0.2] + [1 - 0.6 - 0.2] = 0.4$

   Hence $dis_{\mu^+}(x,z) > dis_{\mu^+}(y,x) + dis_{\mu^+}(y,z)$ which violates the triangle inequality condition required for a metric.

Therefore, this notion of distance does not define a metric on the bipolar fuzzy graph $G = \left(\mu, \tilde{R}\right)$.

Proposition 4.6.1

Let $G=(\mu,\tilde{R})$ be a bipolar fuzzy graph and $\rho: x=x_0,x_1,...,x_n=y$ be a path from x to y. Define the length of ρ as the sum of the reciprocal of the strength of the pairs $(x_{i-1},x_i), i=1,2,...,n$

$$\ell_{\mu^+}(\rho)=\sum_{i=1}^{n}\frac{1}{\tilde{R}^+(x_{i-1},x_i)}$$
$$\ell_{\mu^-}(\rho)=\sum_{i=1}^{n}\left|\frac{1}{\tilde{R}^-(x_{i-1},x_i)}\right| \tag{4.31}$$

If $n=0$, then $\ell_{\mu^+}(\rho)=0$ and $\ell_{\mu^-}(\rho)=0$

If $n\geq 1$, then $\ell_{\mu^+}(\rho)\geq 1$ and $\ell_{\mu^-}(\rho)\geq 1$ since $\tilde{R}^+(x_{i-1},x_i)>0$ and $\tilde{R}^-(x_{i-1},x_i)>-1\ \forall i$

It follows then that for any 2 nodes, we have the $\mu^+-\text{distance}$ $(\text{resp. }\mu^- -\text{distance})$, $d_{\mu^+}(x,y)\left(\text{resp. } d_{\mu^-}(x,y)\right)$ defined as;

$$d_{\mu^+}(x,y)=\min\left(\ell_{\mu^+}(\rho)\right) \text{ and } d_{\mu^-}(x,y)=\min\left(\ell_{\mu^-}(\rho)\right) \tag{4.32}$$

Theorem 4.6.2

$d_{\mu^+}(x,y)\left(\text{resp. } d_{\mu^-}(x,y)\right)$ as defined in proposition 4.5.1 is a metric on the bipolar fuzzy graph $G=(\mu,\tilde{R})$.

a) **Nonnegative**: $d_{\mu^+}(x,y)=\min\left(\ell(\rho)\right)=0 \Rightarrow n=0$. $\ell(\rho)=0$ if $n=0$ which implies $x=y$. $\left(\text{resp. for } d_{\mu^-}(x,y)\right)$

   Also if $n\geq 1$ then $x\neq y$ and $d_{\mu^+}(x,y)=\min\left(\ell_{\mu^+}(\rho)\right)\geq 1$. Hence $d_{\mu^+}(x,y)$ is nonnegative

b) **Symmetric**: path reversal preserves path strength hence

$$d_{\mu^+}(x,y)=\min\left(\ell_{\mu^+}(\rho)\right)=\sum_{i=1}^{n}\frac{1}{\tilde{R}^+(x_{i-1},x_i)}=\sum_{i=1}^{n}\frac{1}{\tilde{R}^+(x_i,x_{i-1})}=d_{\mu^+}(y,x) \text{ and}$$
$$d_{\mu^-}(x,y)=\min\left(\ell_{\mu^-}(\rho)\right)=\sum_{i=1}^{n}\left|\frac{1}{\tilde{R}^-(x_{i-1},x_i)}\right|=\sum_{i=1}^{n}\left|\frac{1}{\tilde{R}^-(x_i,x_{i-1})}\right|=d_{\mu^-}(y,x) \tag{4.33}$$

c) **Triangle property**:

Let $\rho : x = x_0, x_{1,} ..., x_n = z$ be a path from x to z and suppose $y = x_j, j < n$

$$d_{\mu^+}(x,z) = \min\left(\ell_{\mu^+}(\rho)\right) = \sum_{i=1}^{n} \frac{1}{\tilde{R}^+(x_{i-1}, x_i)} \le \sum_{i=1}^{j} \frac{1}{\tilde{R}^+(x_{i-1}, x_j)} + \sum_{i=j}^{n} \frac{1}{\tilde{R}^+(x_j, x_n)} = d_{\mu^+}(x,y) + d_{\mu^+}(y,z)$$

Hence $d_{\mu^+}(x,z) \le d_{\mu^+}(x,y) + d_{\mu^+}(y,z)$

Same argument shows that $d_{\mu^-}(x,z) \le d_{\mu^-}(x,y) + d_{\mu^-}(y,z)$ (4.34)

In summary, we have shown that Definition 4.6.2 introduces the concept of distance in a bipolar fuzzy graph by describing it as the length of the shortest path between two nodes. This definition forms the foundational basis for determining whether such a distance can serve as a valid metric in the bipolar fuzzy framework. To evaluate this, Corollary 4.6.1 examines the proposed distance function against the standard axioms of a metric – non-negativity, symmetry, and the triangle inequality. Through detailed verification and a counterexample, the corollary demonstrates that the defined distance fails to satisfy the triangle inequality and therefore does not qualify as a true metric. Building on this finding, Proposition 4.6.1 refines the earlier formulation by redefining the length of a path in terms of the reciprocal of pairwise strengths. As shown, this ensures that the measure better reflects the underlying structure of the bipolar fuzzy graph. This revised formulation corrects the gaps identified in the corollary and provides a consistent basis for distance computation. Consequently, Theorem 4.6.2 shows that that the improved distance measure proposed in Proposition 4.6.1 satisfies all the properties of a metric thereby confirming that it defines a valid bipolar fuzzy metric space.

4.6.4 Degree of Connectedness

The degree of connectedness of u and v is the weight of the strongest path from x to y. Similarly, we have both the degree of connectedness with respect to the positive membership degree of the vertices in the path ρ from u to v and the degree of connectedness with respect to the negative membership degree of the vertices in the path ρ from u to v, i.e.,

We can then define the degree of $\mu^+ - conectedness\left[resp.\ \mu^- - connectedness\right]$ of u and v, by

$$c_{\mu^+}(u,v) = \max_\mu S_{\mu^+}(\rho)$$
$$c_{\mu^-}(u,v) = \min_\mu S_{\mu^-}(\rho) \quad (4.35)$$

where max and min is taken over all paths from u to v.

Theorem 4.6.3

Let $x, y \in A$. Then

i) $C_{\mu^+}(x,x) = \mu^+(x)$ $\left(\text{resp. } C_{\mu^-}(x,x) = \mu^-(x)\right)$ and

ii) $C_{\mu^+}(x,y) = C_{\mu^+}(y,x)$ $\left(\textit{resp. } C_{\mu^-}(x,y) = C_{\mu^-}(y,x)\right)$ (4.36

Proof

i) Any path ρ x to x from passes through x. Therefore $S_{\mu^+}(\rho) = \min\left\{\tilde{R}(x_{i-1}, x_i)\right\} \leq \mu^+(x)$. On the other hand, x is itself a path of length 0, for which $S_{\mu^+}(\rho) = \mu^+(x)$. Hence $C_{\mu^+}(x,x) = \max\left(S_{\mu^+}(\rho)\right) = \mu^+(x)$. The same argument shows $C_{\mu^-}(x,x) = \mu^-(x)$

ii) Path reversal preserves path strength. Therefore,

$$C_{\mu^+}(x,y) = C_{\mu^+}(y,x) \ \left(\textit{resp. } C_{\mu^-}(x,y) = C_{\mu^-}(y,x)\right) \quad (4.37)$$

Now, since A is a bipolar fuzzy subset, then we have $\mu^+ : X \to [0,1]$ and $\mu^- : X \to [-1,0]$. If $S = \mu^{-1}(1) = \left\{x \middle| x \in X, \mu^+(x) = 1, \mu^-(x) = 0\right\}$, then $S_\mu(\rho) = 1 \ \textit{iff} \ \forall \ x_i \in \rho, x_i \in S$. Further $c_\mu(u,v) = 1 \ \textit{iff} \ u$ and v are connected in S. The degree of connectedness generalizes crisp sense. If $X \subset \mathbb{Z}^n$ is a digital image, then we can define the degree of connectedness of X by $c_\mu(X) = \min_{u,v \in X} c_\mu(u,v)$

Proposition 4.6.2

Let $u, v \in \mu$. Then

$$c_{\mu^+}(u,v) \le \min\{\mu^+(u), \mu^+(v)\}$$
$$c_{\mu^-}(u,v) \le \max\{\mu^-(u), \mu^-(v)\}$$

Proof

The $\mu^+ - \text{strength}$ of any path $\rho : u = v_o, v_1, ..., v_n = v$ is given by

$$S_{\mu^+}(\rho) = \min\{R^+(v_{i-1}, v_i) | 1 \le i \le n\} \le \min\{\mu^+(v_0), \mu^+(v_n)\} = \min\{\mu^+(u), \mu^+(v)\}$$

$$c_{\mu^+}(u,v) \le \min\{\mu^+(u), \mu^+(v)\} \qquad (4.38)$$

Similarly the $\mu^- - \text{strength}$ is given by

$$S_{\mu^-}(\rho) = \max\{R^-(v_{i-1}, v_i) | 1 \le i \le n\} \le \max\{\mu^-(v_0), \mu^-(v_n)\} = \max\{\mu^-(u), \mu^-(v)\}$$

$$c_{\mu^-}(u,v) \le \max\{\mu^-(u), \mu^-(v)\} \qquad (4.39)$$

$c_\mu(u,v)$ is a bipolar fuzzy relation on the bipolar fuzzy set A

Proposition 4.6.3

Let $u,v \in \mu$. Then

$$c_{\mu^+}(u,v) \le \min\{\mu^+(u), \mu^+(v)\}$$
$$c_{\mu^-}(u,v) \le \max\{\mu^-(u), \mu^-(v)\}$$

Proof

The $\mu^+ - \text{strength}$ of any path $\rho : u = v_o, v_1, ..., v_n = v$ is given by

$$S_{\mu^+}(\rho) = \min\{R^+(v_{i-1}, v_i) | 1 \le i \le n\} \le \min\{\mu^+(v_0), \mu^+(v_n)\} = \min\{\mu^+(u), \mu^+(v)\}$$

$$c_{\mu^+}(u,v) \le \min\{\mu^+(u), \mu^+(v)\} \qquad (4.40)$$

Similarly the μ^{-} − strength is given by

$$S_{\mu^{-}}(\rho)=\max\left\{R^{-}(v_{i-1},v_i)\middle|1\le i\le n\right\}\le\max\left\{\mu^{-}(v_0),\mu^{-}(v_n)\right\}=\max\left\{\mu^{-}(u),\mu^{-}(v)\right\}$$

$$c_{\mu^{-}}(u,v)\le\max\left\{\mu^{-}(u),\mu^{-}(v)\right\} \tag{4.41}$$

$c_{\mu}(u,v)$ is a bipolar fuzzy relation on the bipolar fuzzy set A.

4.7 Connectedness on Bipolar Fuzzy Digital Images

As previously mentioned, connectedness is a very central concept in image processing and computer vision. It is employed in various image processing operations including segmentation algorithms and labeling of connected components. The concept of connectedness in fuzzy and bipolar fuzzy digital images has the potential to extend the boundaries of applications of image processing algorithms. The underlying foundational notions that can make the aforementioned possible are explored in the following results.

Lemma 4.7.1

Let $X\subset\mathbb{Z}^n$ be a digital image. Then $c_{\mu}(X)\le\min\left\{\mu(u):u\in X\right\}$.

Proof

If $\mu(u)=\mu(v)=\{1,0\}$ then u and v are connected *iff* $\exists$ a path ρ from u to v $u'\in(u,v)$, such that for any $\mu(u')=\{1,0\}$. Two point u and v are connected in A *iff* they are connected in $S=\left\{\mu^{-1}(\{1,0\})\right\}$. However, points can be connected in μ and fail to be connected in S. Suppose $\mu(u)=\{0,0\}$, then u is connected to any v with $c_{\mu}(u,v)=0$.

are not connected in μ.

Proposition 4.7.1

Let $u,v\in\mu$. Then,

$$c_{\mu^{+}}(u,v)\le\min\left\{\mu^{+}(u),\mu^{+}(v)\right\}$$
$$c_{\mu^{-}}(u,v)\le\max\left\{\mu^{-}(u),\mu^{-}(v)\right\}$$

Proof

The μ^+-strength of any path $\rho: u = v_o, v_1, ..., v_n = v$ is given by

$$S_{\mu^+}(\rho) = \min\left\{R^+(v_{i-1}, v_i)\middle|1 \le i \le n\right\} \le \min\left\{\mu^+(v_0), \mu^+(v_n)\right\} = \min\left\{\mu^+(u), \mu^+(v)\right\}$$

$$c_{\mu^+}(u,v) \le \min\left\{\mu^+(u), \mu^+(v)\right\}. \tag{4.42}$$

Similarly the μ^- $-$strength is given by

$$S_{\mu^-}(\rho) = \max\left\{R^-(v_{i-1}, v_i)\middle|1 \le i \le n\right\} \le \max\left\{\mu^-(v_0), \mu^-(v_n)\right\} = \max\left\{\mu^-(u), \mu^-(v)\right\}$$

$$c_{\mu^-}(u,v) \le \max\left\{\mu^-(u), \mu^-(v)\right\}. \tag{4.43}$$

$c_\mu(u,v)$ is a bipolar fuzzy relation on the bipolar fuzzy set A

Proposition 4.7.2

The degree of connectedness $C_{\mu^+}(x,y)\left(\text{resp. } C_{\mu^-}(x,y)\right)$ is a reflexive and symmetric relation but not necessarily transitive, i.e. $\mu^+ - conectedness\left[resp.\ \mu^- - connectedness\right]$ is not an equivalence relation.

Proof

From proposition 1 above;

Reflexivity: $C_{\mu^+}(x,x) = \mu^+(x)$ $\left(\text{resp. } C_{\mu^-}(x,x) = \mu^-(x)\right)$ and

Symmetry: $C_{\mu^+}(x,y) = C_{\mu^+}(y,x)$ $\left(resp.\ C_{\mu^-}(x,y) = C_{\mu^-}(y,x)\right)$

Transitivity: Suppose $x, y, z \in A$ and $\mu^+(x) = \mu^+(z) = 1$ and $\mu^+(y) < 1$ then (x,y) are connected since $S_{\mu^+}(\rho) = \min\left(\mu^+(x), \mu^+(y)\right) = \mu^+(y)$. Similarly (y,z) are connected since $S_{\mu^+}(\rho) = \min\left(\mu^+(y), \mu^+(z)\right) = \mu^+(y)$. However (x,z) are not connected since $S_{\mu^+}(\rho) \ne \min\left(\mu^+(x), \mu^+(z)\right)$.

$C_{\mu^+}(x,y)\left(\text{resp. } C_{\mu^-}(x,y)\right)$ is not an equivalence relation.

Just like in fuzzy connectedness, this concept of bipolar fuzzy connectedness is not an equivalence relation. Nevertheless, it remains a useful relation since the analogous notion of 'connected components' may be defined in the bipolar fuzzy setting. In the next section, this definition is explored and the accompanying properties are discussed.

4.8 Connected Components in Bipolar Fuzzy Graphs

To define connected components in a bipolar fuzzy graph, both positive and negative edges must be considered during graph traversal. The process involves moving along positive edges in one direction while following negative edges in the opposite direction. In doing so, we ensure a complete representation of the underlying relationships. As a result, analysing bipolar fuzzy graphs requires incorporating both fuzzy memberships and bipolar weights in order to create a more nuanced understanding of connectivity within the system. In this section, the concepts of fuzzy components are extended by defining their analogous versions in bipolar fuzzy graphs. These concepts are plateaus, tops and bottoms.

4.8.1 Plateau

Plateaus represent regions of the bipolar fuzzy graph where elements exhibit a stable degree of membership, both positively and negatively, indicating areas of relatively neutral or balanced interaction.

A $\mu^+ - \text{plateau}\left(\text{resp. } \mu^- - \text{plateau}\right)$ in a bipolar fuzzy subset μ is a maximal $\mu^+ - \text{connected}\left(\text{resp. } \mu^- - \text{connected}\right)$ connected of subset A on which $\mu^+\left(\text{resp. } \mu^-\right)$ has a constant value. That is, $A \subseteq X$ is a plateau if

a) A is $\mu^+ - \text{connected}\left(\text{resp. } \mu^- - \text{connected}\right)$

b) $\forall x, y \in A,\ \mu^+(x) = \mu^+(y)\left(\text{resp. } \mu^-(x) = \mu^-(y)\right)$

c) For all pairs of adjacent vertices such that $x \in A,$ and $y \notin A,$ $\mu^+(x) \neq \mu^+(y)$ $\left(\text{resp. } \mu^-(x) \neq \mu^-(y)\right)$

Note: i) If A is both a $\mu^+ - \text{plateau}$ and a $\mu^- - \text{plateau}$ then we say that A is a $\mu - \text{plateau}$

ii) Any $x \in A$ can only belong to one and only one $\mu^+ - \text{plateau}\left(\text{resp. } \mu^- - \text{plateau}\right)$.

4.8.2 Tops

Tops are defined as the regions where the positive membership function reaches its maximum value, signifying areas of strong support or presence.

A $\mu^+ - \text{plateau}\left(\text{resp. } \mu^- - \text{plateau}\right)$ is called a $\mu^+ - \text{Top}\left(\text{resp. } \mu^- - \text{Top}\right)$ if whenever $x \in A,$ and $y \notin A,$ $\mu^+(x) > \mu^+(y)$ $\left(\text{resp. } \mu^-(x) < \mu^-(y)\right)$. If A is both a $\mu^+ - \text{Top}$ and a $\mu^- - \text{Top}$ then we say that A is a $\mu - \text{Top}$

A $\mu^+ - \text{plateau}\left(\text{resp. } \mu^- - \text{plateau}\right)$ is called a $\mu^+ - \text{Bottom}\left(\text{resp. } \mu^- - \text{Bottom}\right)$ if whenever $x \in A,$ and $y \notin A,$ $\mu^+(x) < \mu^+(y)$ $\left(\text{resp. } \mu^-(x) > \mu^-(y)\right)$. If A is both a $\mu^+ - \text{Bottom}$ and a $\mu^- - \text{Bottom}$ then we say that A is a $\mu - \text{Bottom}$

Definition 4.8.1

Suppose A is a $\mu - \text{Top}$, then we may associate to A the following sets;

a)
$$\Omega_{A^+} = \left\{x \in A \mid \exists \rho : x = x_o, x_1, \ldots, x_n = y \text{ with } \mu^+(x_{i-1}) \le \mu^+(x_i)\right\}$$
$$\Omega_{A^-} = \left\{x \in A \mid \exists \rho : x = x_o, x_1, \ldots, x_n = y \text{ with } \mu^-(x_{i-1}) \ge \mu^+(x_i)\right\}$$

b)
$$\Pi_{A^+} = \left\{x \in A \mid \exists \rho : x = x_o, x_1, \ldots, x_n = y \text{ with } \mu^+(x) \le \mu^+(x_i) \le \mu^+(y)\right\}$$
$$\Pi_{A^-} = \left\{x \in A \mid \exists \rho : x = x_o, x_1, \ldots, x_n = y \text{ with } \mu^-(x) \ge \mu^-(x_i) \ge \mu^-(y)\right\}$$

c)
$$\Sigma_{A^+} = \left\{x \in A \mid \exists \rho : x = x_o, x_1, \ldots, x_n = y \text{ with } \mu^+(x) \le \mu^+(x_i)\right\}$$
$$\Sigma_{A^-} = \left\{x \in A \mid \exists \rho : x = x_o, x_1, \ldots, x_n = y \text{ with } \mu^-(x) \ge \mu^-(x_i)\right\}$$

Theorem 4.8.1

Let A be a $\mu - \text{Top}$, then

$$A^+ \subseteq \Omega_{A^+} \subseteq \Pi_{A^+} \subseteq \Sigma_{A^+}$$
$$A^- \subseteq \Omega_{A^-} \subseteq \Pi_{A^-} \subseteq \Sigma_{A^-}$$

From the definitions above, a point x belongs in Ω_{A^+} ($resp\ \Omega_{A^-}$) if there exists a monotonically nondecreasing bipolar fuzzy path from x to A. Consequently, it is not possible to have a peak higher than the $Top\ A$. By the same argument, if a point x belongs in Π_{A^+} ($resp\ \Pi_{A^-}$) or x belongs in $\sum_{A^+}$ ($resp \sum_{A^-}$) then there cannot exist a peak higher than the $Top\ A$ between x and A.

Corollary 4.8.1

Let A and B be two $\mu-\text{Top}$. Then A and B cannot be adjacent to each other.

Proof

If they have the same height, then A and B are a single $\mu-\text{Top}$. If A and B have different heights, then the shorter one cannot be a $\mu-\text{Top}$

Proposition 4.8.2

Let A^+ be a $\mu^+-\text{Top}$ then Σ_{A^+} is essentially the set of all points of X that are connected to points *iff* $\Sigma_{A^+}=\left\{x\in A \mid \exists\rho : x=x_o,x_1,...,x_n=y \text{ with } \mu^+\left(x\right)\leq\mu^+\left(x_i\right)\right\}$ of A^+

Proof

Let X, a nonempty set of integer coordinate points endowed with a k-adjacency relation, be connected to $y\in A^+$. Then there exists a path ρ from x to y such that for all points x_i on the path ρ, $\mu^+(x_i)\geq\min\left(\mu^+(x),\mu^+(y)\right)$.

If $\mu^+(x)>\mu^+(y)$ then $x\notin A^+$ and $\mu^+(x)\geq\mu^+(y)\ \forall x_i$ on the path ρ. But form the proposition above this is not possible since the path ρ must pass through a point y' adjacent to A^+ but not in A^+. Therefore, it is mandatory that $\mu^+(y')<\mu^+(A)$ hence $\mu^+(x)<\mu^+(y)$ and $\mu^+(x_i)\geq\mu^+(x)$ $\forall x_i$ on the path ρ

Conversely, if $x\in\Sigma_{A^+}$ then $\mu^+(x)\leq\mu^+(A)$ by above proposition. Therefore, there exists a path ρ from x to a point y of A^+ such that $\forall x_i$ on the path ρ, then $\mu^+(x_i)\geq\mu^+(x)=\min\left(\mu^+(x),\mu^+(y)\right)$, implying that x is connected to y

In this section, the concepts of fuzzy components have been expanded by defining their analogous counterparts in bipolar fuzzy graphs. In traditional fuzzy graphs, components are typically classified based on similarity or proximity. However, in bipolar fuzzy graphs, these components have been defined to reflect both positive and negative influences within the same structure.

4.9 Bipolar Fuzzy Surroundness

The extent to which a digital structure is surrounded is related to how much the path must change direction in order to reach the boundary without intersections. This measure reflects the complexity of the structure's perimeter, indicating how frequently a path must adjust its direction to avoid crossing through internal points. The more intricate the boundary, the greater the number of directional changes required to navigate around the structure without interference. This concept is very useful when it comes to understanding the topological properties of digital structures. It also has applications in many fields such as image processing and pattern recognition.

Definition 4.9.1

Let $A=\left(A^{+},A^{-}\right),B=\left(B^{+},B^{-}\right),C=\left(C^{+},C^{-}\right)$ be bipolar fuzzy subsets of X. B is said to separate A from C if $\forall x\in X$ and all paths ρ from x to y there exists a point y on the path ρ such that

$$B^{+}\left(y\right)\geq \min\left(A^{+}\left(x\right),C^{+}\left(z\right)\right) \text{ and}$$
$$B^{-}\left(y\right)\leq \max\left(A^{-}\left(x\right),C^{-}\left(z\right)\right)$$

In this formulation, B surrounds A if it separates A from the border of X

Theorem 4.9.1

The relation B surrounds A is a weak partial order. i.e., the relation is reflexive, antisymmetric and Transitive

Proof

Let $A=\left(A^{+},A^{-}\right),B=\left(B^{+},B^{-}\right),C=\left(C^{+},C^{-}\right)$ be bipolar fuzzy subsets of X. Then,

a) Reflexivity: A surrounds A
b) Transitivity: Let $x\in X$, and ρ be any path from x to the border of B. If B surrounds C,

then there exists a point $y \in X$ on the path ρ such that $B^{+}(y) \geq C^{+}(x)$ and if A surrounds B, then similarly there exists a point $y \in X$ on the path ρ such that $A^{+}(y) \geq B^{+}(x)$ and $A^{-}(y) \leq B^{-}(x)$. Therefore $A^{+}(y) \geq C^{+}(x)$ and $A^{-}(y) \leq C^{-}(x)$ hence A surrounds C.

c) Antisymmetry: suppose A surrounds B and B surrounds A. Then to proof that the relation is antisymmetric, it is enough to show that $A \wedge B$ surrounds both A and B. Now let ρ be any path from x to the border of B and y be the last point on the path such that $B^{+}(y) \geq A^{+}(x)$ and $B^{-}(y) \leq A^{-}(x)$. Since A surrounds B then there exists a point y' on the path ρ beyond y (or possibly y itself) such that $A^{+}(y') \geq B^{+}(y)$ and $A^{-}(y') \leq B^{-}(y)$. Similarly, since B surrounds A, then there exists a point y'' on the path ρ beyond y' (or possibly y' itself) such that $B^{+}(y'') \geq A^{+}(y') \geq A^{+}(x)$ and $B^{-}(y'') \leq A^{-}(y') \leq A^{-}(x)$. From the choice of, then $y = y' = y''$ so that $A^{+}(y) \wedge B^{+}(y) \geq A^{+}(x)$ and $A^{-}(y) \wedge B^{-}(y) \leq A^{-}(x)$. Since x is arbitrary, then $A \wedge B$ surrounds A and similarly surrounds B

The relation is a weak partial order.

4.10 Applicability/applications of the theory in image analysis

The results generated on the properties of ‘connectedness’ and ‘surroundness’ with respect to bipolar fuzzy sets generalize some of the standard results of digital images in ordinary subsets. These are of some practical interest in connection with digital picture segmentation as proposed in this section.

4.10.1 Conceptual Framework for Bipolar Fuzzy Similarity Measures

In the context of fuzzy image representation, each pixel intensity $I(p_i)$ in an image is transformed into a single-valued membership function. To extend this idea into the bipolar fuzzy context, each pixel will be assigned two membership values. To achieve this, we will use the following transformation

$$B(p_i) = \left(\mu^{+}(p_i), \mu^{-}(p_i)\right) \tag{4.44}$$

where $\mu^+(p_i)$ and $\mu^-(p_i)$ represent the degree of relevance to the desired feature and undesired features respectively.

For grayscale images, the bipolar fuzzy memberships functions may be defined as

$$\mu^+(p_i) = \frac{I(p_i) - \min(I)}{\max(I) - \min(I)} \tag{4.45}$$

where;

i) $I(p_i)$ is the grayscale intensity of the pixel p_i
ii) $\min(I)$ is the lowest intensity in the image (typically 0 for black intensity)
iii) $\max(I)$ is the highest intensity value in the image (usually 255 for white intensity)
iv) $\mu^+(p_i) \in [0,1]$

This proposed transformation can be shown by taking $I(p_i) = 200$. In this case

$\mu^+(p_i) = \frac{200-0}{255-0} = 0.784$ (a pixel with higher intensity will have a higher positive membership)

The negative membership function $\mu^-(p_i)$ determines how strongly a pixel belongs to the background, noise, or undesired features. By definition this will be given by

$$\mu^-(p_i) = -(1 - \mu^+(p_i)) \in [-1, 0] \tag{4.46}$$

This formulation ensures that brighter pixels have higher μ^+ - values and lower μ^- values and vice versa.

In example above,

if $\mu^+(p_i) = 0.784$ then

$$\mu^-(p_i) = -(1 - 0.784) = -0.216 \tag{4.47}$$

This corresponds perfectly with the fact that since the positive membership measures the presence of a desired feature, the negative membership naturally represents the absence of that feature. Therefore, a pixel of higher intensity will have a lower negative membership.

4.10.2 Bipolar Fuzzy Similarity Measures

To compare two bipolar fuzzy images B_1 and B_2 we extend classical fuzzy similarity measures to account for both positive and negative memberships. The general form of a bipolar fuzzy similarity measure may be expressed as:

$$S_B\left(B_1, B_2\right) = \alpha S^+\left(F_1, F_2\right) - \beta S^-\left(F_1, F_2\right) \tag{4.48}$$

where,

i) $S^+\left(F_1, F_2\right)$ is the similarity between positive memberships.

ii) $S^-\left(F_1, F_2\right)$ is the similarity between negative memberships.

iii) α, β are weighting factors that determine the importance of positive and negative similarities.

In Section 4.2.2, a characterisation of Bipolar Jaccard similarity for Bipolar fuzzy sets was proposed. This was defined for two bipolar fuzzy sets A and B as

$$S\left(A, B\right) = \sum_{i=1}^{n} \left(\frac{A^+\left(x_i\right) \wedge B^+\left(x_i\right), A^-\left(x_i\right) \wedge B^-\left(x_i\right)}{A^+\left(x_i\right) \vee B^+\left(x_i\right), A^-\left(x_i\right) \vee B^-\left(x_i\right)} \right) \tag{4.49}$$

This characterisation is equivalent to

$$S\left(A, B\right) = \frac{\sum \min\left(A^+\left(x_i\right), B^+\left(x_i\right)\right) - \sum \min\left(A^-\left(x_i\right), B^-\left(x_i\right)\right)}{\sum \max\left(A^+\left(x_i\right), B^+\left(x_i\right)\right) + \sum \max\left(A^-\left(x_i\right), B^-\left(x_i\right)\right)} \tag{4.50}$$

In this formulation, the numerator captures the overlap in positive memberships while taking care of dissimilarities in negative memberships, while the denominator normalizes the measure to endure that $S\left(A, B\right) \in \left[-1, 1\right]$.

4.10.3 Bipolar Jaccard Similarity for Grayscale Images Using Min-Max Normalization

In the next section, we present a systematic step-by-step approach for computing the bipolar Jaccard similarity between two grayscale images. The utility of this approach can be seen in domains where intensity differences carry critical information and therefore the bipolar approach is potentially more effective in smoothening out minor variations thereby making similarity computation more robust against unwanted variations in pixel intensity.

This step-by-step approach is presented as a means of validating the effectiveness of the proposed Bipolar Fuzzy Jaccard Similarity, demonstrating its ability to enhance similarity computations

Suppose we have to 4×4 grayscale images as below

Image 1

$$A = \begin{matrix} 50 & 80 & 120 & 200 \\ 30 & 90 & 130 & 210 \\ 60 & 100 & 140 & 220 \\ 70 & 110 & 150 & 230 \end{matrix}$$

Image 2

$$B = \begin{matrix} 55 & 85 & 125 & 205 \\ 35 & 95 & 135 & 215 \\ 65 & 105 & 145 & 225 \\ 75 & 115 & 155 & 235 \end{matrix}$$

Step 1: Compute the minimum and maximum pixel intensities

Find the minimum and maximum pixel values across both images in order to normalize the images

$$\begin{aligned} \min(A,B) &= \min(30,35) = 30 \\ \max(A,B) &= \max(230,235) = 235 \end{aligned} \tag{4.51}$$

Step 2: Compute positive membership function μ^+ using min-max normalization

For every pixel p_i, then by min-max normalization $\mu^+(p_i) = \dfrac{I(p_i) - \min(I)}{\max(I) - \min(I)}$ q

$$\mu^+(p_i) = \frac{A(p_i) - 30}{235 - 30}$$

In A we have

$$\mu_A^{\ +} = \begin{bmatrix} \frac{50-30}{205} & \frac{80-30}{205} & \frac{120-30}{205} & \frac{200-30}{205} \\ \frac{30-30}{205} & \frac{90-30}{205} & \frac{130-30}{205} & \frac{210-30}{205} \\ \frac{60-30}{205} & \frac{100-30}{205} & \frac{140-30}{205} & \frac{220-30}{205} \\ \frac{70-30}{205} & \frac{110-30}{205} & \frac{150-30}{205} & \frac{230-30}{205} \end{bmatrix}$$

$$\mu_A^{\ +} = \begin{bmatrix} 0.0976 & 0.2440 & 0.4390 & 0.8293 \\ 0 & 0.2926 & 0.4878 & 0.8780 \\ 0.1463 & 0.3415 & 0.5366 & 0.9268 \\ 0.1951 & 0.3902 & 0.5854 & 0.9756 \end{bmatrix}$$

Similarly for B

$$\mu_B^{\ +} = \begin{bmatrix} 0.1220 & 0.2683 & 0.4634 & 0.8537 \\ 0.0244 & 0.3171 & 0.5122 & 0.9024 \\ 0.1707 & 0.3659 & 0.5610 & 0.9512 \\ 0.2195 & 0.4146 & 0.6098 & 1 \end{bmatrix}$$

Step 3: Compute negative membership function μ^-

The negative membership function $\mu^-(p_i)$ is defined by $\mu^-(p_i) = -(1-\mu^+(p_i)) \in [-1,0]$

In A we have

$$\mu_A^{\ -} = \begin{bmatrix} -0.9024 & -0.7561 & -0.5610 & -0.1707 \\ -1 & -0.7073 & -0.5122 & -0.1220 \\ -0.8537 & -0.6585 & -0.4634 & -0.0732 \\ -0.8049 & -0.6098 & -0.4146 & -0.0244 \end{bmatrix}$$

For B

$$\mu_B^{\ -} = \begin{bmatrix} -0.8780 & -0.7317 & -0.5366 & -0.1463 \\ -0.9756 & -0.6829 & -0.4878 & -0.0976 \\ -0.8293 & -0.6341 & -0.4390 & -0.0488 \\ -0.7805 & -0.5854 & -0.3902 & 0 \end{bmatrix}$$

Step 4: Compute bipolar Jaccard similarity

The bipolar Jaccard similarity is given by

$$s(A,B)=\frac{\sum\min\left(A^{+}(x_i),B^{+}(x_i)\right)-\sum\min\left(A^{-}(x_i),B^{-}(x_i)\right)}{\sum\max\left(A^{+}(x_i),B^{+}(x_i)\right)+\sum\max\left(A^{-}(x_i),B^{-}(x_i)\right)} \tag{4.52}$$

It can be verified that

$$s(A,B)=\frac{-0.5584}{16.9884}=-0.077$$

The Bipolar Jaccard Similarity value of -0.077 suggests that the images have slightly more dissimilarities than similarities. This means that while some pixels have similar intensity values, there are still noticeable variations. The negative membership values indicate a higher degree of dissimilarity than similarity (if the similarity score were closer to 1, it would suggest that the images are nearly identical, whereas a score approaching -1 would indicate significant differences between them).

4.10.4 Bipolar Fuzzy Jaccard Similarity Using Sigmoid-Based Membership

Instead of using linear min-max normalization, we define sigmoid-based fuzzy membership functions by

$$\mu^{+}(p)=\frac{1}{1+e^{-k(I(p)-T)}}$$
$$\mu^{-}(p)=-\left(1-\mu^{+}(p)\right) \tag{4.54}$$

where;

i) $I(p)$ is the pixel intensity

ii) T is the threshold intensity

iii) k controls the steepness of the function (chosen as 0.1 for this example)

Suppose we have to 4×4 grayscale images as below

Image 1 **Image 2**

$$A = \begin{matrix} 50 & 80 & 120 & 200 \\ 30 & 90 & 130 & 210 \\ 60 & 100 & 140 & 220 \\ 70 & 110 & 150 & 230 \end{matrix} \qquad B = \begin{matrix} 55 & 85 & 125 & 205 \\ 35 & 95 & 135 & 215 \\ 65 & 105 & 145 & 225 \\ 75 & 115 & 155 & 235 \end{matrix}$$

Then by computing this in Python using the Bipolar fuzzy Jaccard formula,

$$s(A,B) = \frac{\sum \min\left(A^{+}(x_i), B^{+}(x_i)\right) - \sum \min\left(A^{-}(x_i), B^{-}(x_i)\right)}{\sum \max\left(A^{+}(x_i), B^{+}(x_i)\right) + \sum \max\left(A^{-}(x_i), B^{-}(x_i)\right)} \tag{4.53}$$

we obtain a Bipolar Fuzzy Jaccard Similarity -0.1120

See Appendix F for the Full Python implementation of the Bipolar Fuzzy Jaccard Similarity algorithm.

The BJS with sigmoid transformation is lower (-0.112) than with the min-max transformation (-0.077) implying the two functions assigns bipolar fuzzy values differently. This difference could be as a result of

i) Min-max preserves relative pixel differences, making it more sensitive to slight variations.
ii) Sigmoid-based membership compresses extreme values, emphasizing mid-range intensities, which affects how differences in brightness levels contribute to similarity.
iii) The penalty for mismatches (negative membership function) behaves differently in sigmoid-based transformation, leading to a more negative similarity score in some cases

Table 4.2: Key differences between the two approaches

Normalisation Method	Effect on Membership Values	Effect on Similarity Score
Min-Max Normalisation (Linear)	Preserves relative intensity differences	More sensitive to small pixel changes, leading to lower similarity if pixel values differ

Sigmoid-Based Membership (Non-Linear)	Sigmoid-Based Membership (Non-Linear)	Less sensitive to small differences but may yield negative similarity scores due to contrast compression

CHAPTER FIVE
CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction

This chapter presents the concluding remarks and key insights derived from this study. The research concerned the idea of integrating bipolar fuzzy logic into digital topology and their application in image analysis. By enhancing the representation of uncertainty and relationships in digital images using bipolar fuzzy sets, the findings of this study have been shown to have useful implications for image analysis.

5.2 Conclusions

The aim of this research was to address the limitations in digital topology by looking at ways in which fuzzy and bipolar fuzzy metric spaces can be integrated into image analysis. The research identified the gaps that exist in digital topology, and attempted to bridge these gaps. To do this, the study introduced digital fuzzy and bipolar fuzzy metric spaces as an extension of the classical metrics. Below is an evaluation of how each objective was met.

Specific Objective 1

This research formulated fuzzy and bipolar fuzzy metric spaces and established their completeness. In Section 4.2, the study introduced the concept of digital fuzzy metric spaces. The axioms were introduced and were proved to conform to properties of metric spaces. Doing this ensured that what was developed was consistent with what already existed. The bipolar fuzzy metric framework was then further discussed in Section 4.6. In this section, concepts such as path strength, length, and a metric for bipolar fuzzy graphs were defined. We then showed that these results are also consistent with what is expected..

Specific Objective 2

This objective was meant to address the limitation of digital topology when producing models that can be used in environments that are vague and uncertain. To do this, the research developed extensions of properties of fuzzy mathematics into bipolar fuzzy domains. This has ensured that digital topology can then be used to study images that are very uncertain or even blurred. In Section 4.8, the concept of connectedness in bipolar fuzzy digital images was developed and discussed.

This included the properties of connected components, plateaus, Tops and Bottoms, and the degree of connectedness.

Specific Objective 3

This research also developed new measures of similarity, distance, and entropy because these concepts are very critical for analysing digital images. The aim was to improve the foundation sof process that are used in pattern recognition and segmentation accuracy. In Section 4.2, we introduced a bipolar fuzzy similarity measure and this was followed by characterising the Jaccard similarity measure to fit bipolar fuzzy sets in Section 4.2.2. This was done so as to have a better method that can be used to quantifying the nature of fuzziness and bipolarity. The research also defined an entropy measure in Section 4.3 because entropy measure give a more accurate representation of uncertainty in the context of bipolar fuzzy domains. These measures enhance the mathematical precision of image processing models, ensuring more effective feature extraction and classification in applications.

Specific Objective 4

Finally, an important aspect of this work was validation. This was achieved by performing a computation on pixels in grayscale images. In Section 4.10, the study applied bipolar fuzzy Jaccard similarity to grayscale image and used min-max normalisation to show the results. Furthermore, Section 4.10.3 used the sigmoid-based membership functions to compute the bipolar fuzzy Jaccard similarity and these was compared to the result from using the minmax normalisaton. These computational results confirm that the proposed fuzzy and bipolar fuzzy frameworks can indeed enhance the accuracy image processing models.

5.3 Recommendations

The findings presented in Chapter Four proved valuable contributions to the development of bipolar fuzzy metric spaces and how this may be applied to image anlysis. By extending conventional fuzzy metric theory the findings show that we can introduce dual membership functions in order to capture the dual aspects of pixels in images. This advancement provides a more comprehensive framework for handling real-world scenarios that consist of conflicting information. Furthermore, the study illustrates how classical distance and similarity measures can be successfully adapted to bipolar fuzzy environments while maintaining coherence with what is already established in the area of fuzzy and bipolar fuzzy logic.

Building on these results, the following recommendations are proposed to advance the theoretical and applied dimensions of this research:

i. Further work should aim to refine the theoretical foundation of fuzzy and metrics, ensuring mathematical completeness and internal consistency in digital topology frameworks.
ii. Key topological concepts should be systematically adapted and validated within bipolar fuzzy digital spaces to strengthen their applicability in other kinds of models.
iii. Research should continue to focus on improving the similarity measure to make them applicable in diverse areas of application.

5.3.1 Future Research Direction

While this study establishes mathematical foundation for bipolar fuzzy metric spaces, further research is necessary to refine, and expand the scope of application of these concepts. This has been discussed in the preceding section. Some of these field include, artificial intelligence, digital image processing, and uncertainty modeling.

Machine learning (ML) and artificial intelligence (AI) have revolutionised data-driven decision-making and predictive modeling. However, most conventional ML models rely on crisp, well-defined feature spaces, which makes them less effective in scenarios where uncertainty and contradictions are common. The integration of bipolar fuzzy logic into machine learning models can enhance pattern recognition and optimisation AI models. In such cases bipolar fuzzy distances can serve as powerful engineering tools.

Although the theoretical foundations of bipolar fuzzy metric spaces have been developed, their real-world implementation remains limited. Future research could focus on validating these concepts in practical applications, particularly in fields where uncertainty, imprecision, and contradictory data are prevalent, for instance medical diagnostics and health informatics.

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APPENDICES

Appendix A: Abstracts of published work

Advances in Pure Mathematics > Vol.14 No.7, July 2024

Check for updates

Connected Components in Bipolar Fuzzy Digital Plane

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DOI: 10.4236/apm.2024.147031 **PDF** **HTML** **XML** **78** Downloads **328** Views

Abstract

The concepts of connectedness play a critical role in digital picture segmentation and analyses. However, the crisp nature of set theory imposes hard boundaries that restrict the extension of the underlying topological notions and results. Whilst fuzzy set theory was introduced to address this inherent drawback, most human processes are not just fuzzy but also double-sided. Most phenomena will exhibit both a positive side and a negative side. Therefore, it is not enough to have a theory that addresses imprecision, uncertainty and ambiguity; rather, the theory must also be able to model polarity. Hence the study of bipolar fuzzy theory is of potential significance in an attempt to model real-life phenomena. This paper extends some concepts of fuzzy digital topology to bipolar fuzzy subsets including some important basic properties such as connectedness and surroundedness.

Keywords

Fuzzy, Bipolar Fuzzy, Digital Topology

Advances in Pure Mathematics > Vol.15 No.10, October 2025

Check for updates

A Bipolar Fuzzy Approach to Image Segmentation: Enhancing Similarity Measures and Entropy Computation

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DOI: 10.4236/apm.2025.1510034 **PDF** **HTML** **XML** **26** Downloads **109** Views

Abstract

Image segmentation is a fundamental process in digital image analysis, with applications in object recognition, medical imaging, and computer vision. Traditional segmentation techniques often struggle with uncertainty, imprecise boundaries, and misclassified regions due to their inability to effectively model both positive and negative information. This study introduces a bipolar fuzzy-based computational approach that enhances segmentation accuracy by incorporating dual membership functions to represent both the presence and absence of image features. To further improve segmentation robustness, we extend classical similarity measures by formulating a Bipolar Fuzzy Jaccard Similarity, which quantifies both positive and negative membership interactions, leading to more precise region classification. Additionally, a novel Bipolar Rényi Entropy (BRE) measure is developed to capture uncertainty in segmentation by integrating bipolar fuzzy probability distributions, allowing for adaptive sensitivity to dominant and rare features. Experimental validation on grayscale image datasets demonstrates the superiority of the proposed approach over conventional fuzzy and graph-based segmentation methods, particularly in applications requiring high precision, such as medical imaging and AI-driven pattern recognition. The integration of bipolar fuzzy similarity and entropy measures provides a powerful computational framework for more accurate and interpretable image segmentation.

Keywords

Fuzzy, Bipolar Fuzzy, Similarity Measure, Entropy

Appendix B: Useful Definitions in Fuzzy Set theory

Types of membership functions

Membership functions can either be chosen arbitrarily by the user based on the user's need or be designed using machine learning methods **(**Bellman and Giertz 1973). Zadeh 1965 proposed a series of membership functions that could be classified into two groups, linear and the curved or nonlinear membership functions. There are different shapes of membership functions as seen in Zimmerman 2001

Triangular membership

A triangular membership function is specified by three parameters a,b,c where a is the lower boundary and c the upper boundary where elements have a membership degree of 0 and b is the center where elements have a membership degree of 1

$$\mu_A(x)=\begin{cases}0 & \text{if } x\le a\\ \dfrac{x-a}{b-a} & \text{if } a\le x\le b\\ \dfrac{c-x}{c-b} & \text{if } b\le x\le c\\ 0 & \text{if } x\ge c\end{cases}$$

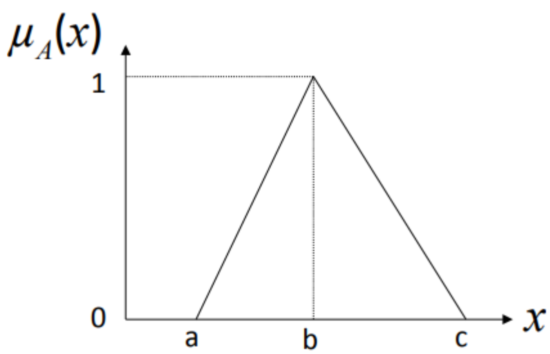

Figure A1 triangular membership function

Trapezoidal membership

A trapezoidal membership function is specified by four parameters a,b,c,d as follows

$$\mu_A(x)=\begin{cases}0 & \text{if } x\le a\\ \dfrac{x-a}{b-a} & \text{if } a\le x\le b\\ 1 & \text{if } b\le x\le c\\ \dfrac{d-x}{d-c} & \text{if } c\le x\le d\\ 0 & \text{if } d\le x\end{cases}$$

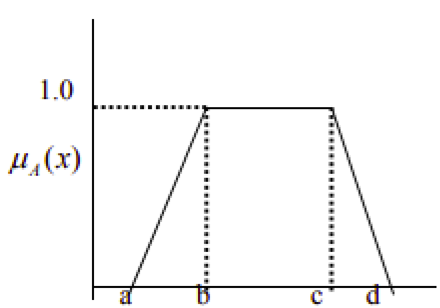

Figure A2: A trapezoidal membership function

Gaussian Membership

The Gaussian membership function is usually represented as

$$\mu_A(x, c, s, m) = \exp\left[-\frac{1}{2}\left|\frac{x-c}{s}\right|^m \right]$$, c is the center, s is the width and m is the fuzzification factor.

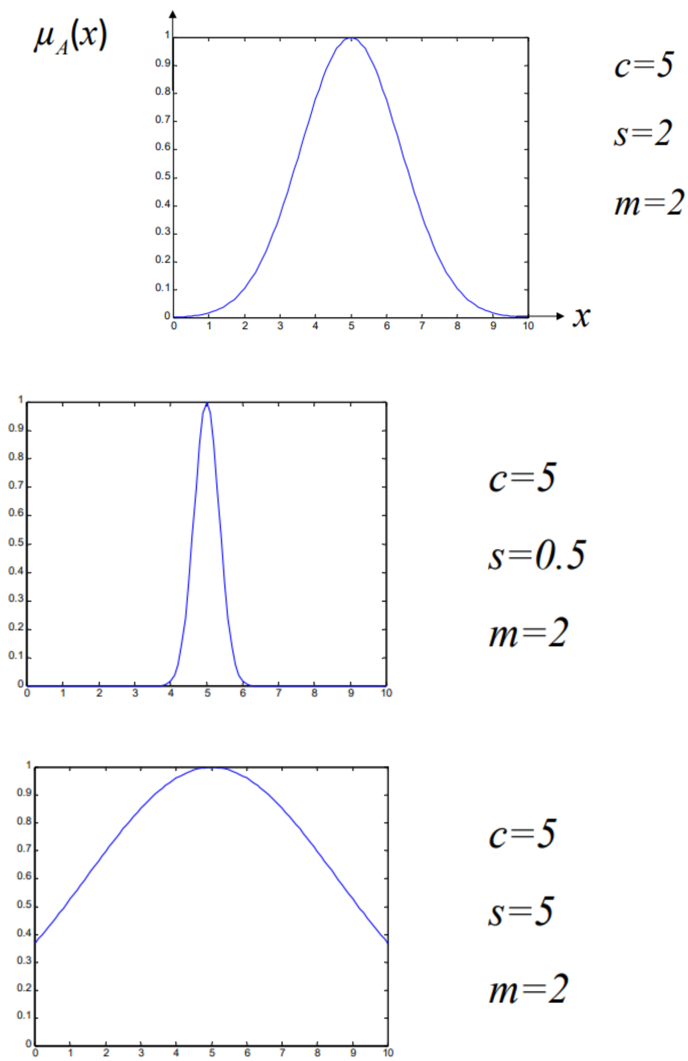

Figure A3: The Gaussian membership function different values of c, s and m

Appendix C: Applications of fuzzy sets in facility location modeling

Over the past decades, network-based facility location modeling has experienced significant advancements (Marianov & Serra, 2004). Traditional facility location models, such as the Simple Plant Location Problem (SPLP), have been widely used due to their mathematical simplicity. However, these models primarily rely on binary inputs, which demand precisely defined decision criteria. This makes them less suited for real-world problems where subjective and qualitative factors play a crucial role (Marianov & Serra, 2004). Many critical social and policy considerations cannot be captured using binary optimisation techniques. Concepts such as:

- How *far* should people reasonably travel to reach essential services?
- How *important* is road quality in determining accessibility?
- Does *social class homogeneity* influence service demand?
- Is it *unfair* to concentrate multiple major facilities in one area?

These factors involve gradual transitions rather than absolute values, making them ideal for a fuzzy sets approach (Zimmermann, 2001). To enhance facility location models, fuzzy integer programming (FIP) can be employed to integrate soft constraints and gradual preferences into decision-making (Zimmermann, 2001). This approach enables policymakers to:

1. Quantify qualitative factors such as accessibility, fairness, and social impact.
2. Model trade-offs between cost-efficiency and social equity.
3. Introduce adaptive constraints that adjust to real-world uncertainties, such as demographic changes or urban expansion.

Appendix D: Application of Intuitionistic Fuzzy Sets in Career Determination

One of the practical applications of Intuitionistic Fuzzy Sets (IFS) is in career determination, as explored by Ejegwa et al., (2014). Their work uses Euclidean distance to assess the proximity of a student's academic performance to a particular career choice.

Conceptual Framework

The challenge in career determination arises from the fact that academic performance alone is not the only determinant of career suitability (Ejegwa et al., 2014). We often classify students based on rigid conditions (e.g., if a student scores above 80% in science subjects, they are recommended for engineering). However, in reality, students' abilities and preferences involve a degree of uncertainty, hesitation, and subjectivity, which can be effectively modeled using IFS (Ejegwa et al., 2014).

Ejegwa et al., (2014) proposed a fuzzy decision-making model where:

- A student's academic performance in various subjects is represented as an intuitionistic fuzzy set (IFS).
- The degree of membership (μ) represents how strongly a student's performance aligns with a given career.
- The degree of non-membership (υ) represents how strongly the student's performance suggests unsuitability for the career.
- The hesitation degree (π) accounts for uncertainty in subject relevance or variations in grading systems.

Mathematical Model

The core of the model is based on the normalized Euclidean distance, which measures the closeness of a student's intuitionistic fuzzy academic profile to an ideal career profile (Ejegwa et al., 2014)..

Step 1: Define the IFS Representation

Each career choice is associated with a set of key subjects that are considered essential for success in that field (Ejegwa et al., 2014). For a given student S their academic performance in a set of n subjects is represented as an IFS:

$$S = \{(x_i, \mu_i, \upsilon_i) : x_i \in X, 0 \le \mu_i + \upsilon_i \le 1\}$$

where:

- x_i represents a subject,
- μ_i is the degree to which the student's score in x_i supports a particular career,
- υ_i is the degree to which the student's score in xix_ixi opposes the career,
- $\pi_i = 1 - \mu_i - \upsilon_i$ represents the hesitation in decision-making.

Each career path is similarly modeled using an IFS based on ideal subject performance expectations.

Step 2: Calculate the Normalized Euclidean Distance

The distance between the student's performance profile and the ideal career profile is computed using:

$$d(S,C) = \sqrt{\sum_{i=1}^{n} \left((\mu_{S,i} - \mu_{C,i})^2 + (\upsilon_{S,i} - \upsilon_{C,i})^2 \right)}$$ where:

- $\mu_{S,i}$, $\upsilon_{S,i}$ are the student's intuitionistic fuzzy membership values for subject xix_ixi,
- $\mu_{C,i}$, $\upsilon_{C,i}$ are the career's expected intuitionistic fuzzy values for the same subject.

This distance $d(S,C)$ determines how well the student's profile aligns with a given career:

- Smaller $d(S,C) \rightarrow$ Higher suitability for the career.
- Larger $d(S,C) \rightarrow$ Lower suitability for the career.

Step 3: Decision-Making

- The career with the smallest distance to the student's IFS profile is suggested as the most suitable career choice.
- If multiple careers have similar distances, additional factors like personal interests, soft skills, and external recommendations can be incorporated using multi-criteria decision-making (MCDM) techniques.

Illustrative Example

Let's consider a student interested in Engineering (Ejegwa et al., 2014). Suppose their performance in Mathematics, Physics, and Chemistry is evaluated as follows:

Subject	μ_S ***(Membership)***	υ_S ***(Non-membership)***	π_S ***(Hesitation)***
Mathematics	0.85	0.10	0.05
Physics	0.75	0.15	0.10
Chemistry	0.80	0.12	0.08

The ideal profile for Engineering might be:

Subject	μ_C ***(Membership)***	υ_C ***(Non-membership)***
Mathematics	0.90	0.05
Physics	0.80	0.10
Chemistry	0.85	0.07

Using the Euclidean distance formula:

$$d(S,C) = \sqrt{(0.85-0.90)^2 + (0.10-0.05)^2 + (0.75-0.80)^2 + (0.15-0.10)^2 + (0.80-0.85)^2 + (0.12+0.07)^2}$$
$$d(S,C) \approx 0.12$$

Since Engineering has the lowest distance score compared to other career options, it is recommended as the best match for the student (Ejegwa et al., 2014).

Appendix E: Axiomatic Digital topology

In contrast to the graph-theoretic approach, which was discussed in Chapter Two of this thesis, the axiomatic approach to digital topology establishes a rigorous mathematical foundation for digital topology by defining axioms that digital spaces must satisfy, analogous to classical topology. This approach was pioneered by Khalimsky, Rosenfeld, and Kovalevsky, who introduced topological structures specifically tailored for discrete and finite digital spaces (Khalimsky, 1971; Rosenfeld, 1979; Kovalevsky, 1989).

In the axiomatic approach, certain subsets of the underlying digital structures are declared to be open sets and are required to fulfill certain axioms, chosen in such a way that the digital structures gets properties which are as close as possible to the properties of usual topology. A locally finite space, most specifically the Alexandroff space makes this possible, since as shown later, the topology of the Connected Ordered Topological Space (COTS) is completely determined by showing the minimal neighborhood of each element of a digital set. The COTS leads to the definition of the Khalimasky topology, (Khalimsky et al., 1991)

From Kovalevsky, (1989), A nonempty set S is called a locally finite space if $\forall x \in S,\ \exists\ U_i \subseteq S, i = \{1, 2, .., n\}$ such that $x \in U_i$. We denote these neighborhoods by U_x

Axioms of digital topology

The following axioms were posited by Kovalevsky, (2006).

Axiom 1: The intersection of two neighborhoods of x is again a neighborhood of x,

$$\forall U_x, V_x \subseteq S, \left(U_x \cap V_x\right)_x$$

Since the space is locally finite, there exists the smallest neighborhood of x, which is the intersection of all the neighborhoods of x. This smallest neighborhood of x is denoted by $SN(x)$

Axiom 2: There exists space elements which have in their SN more than one element. If $y \in SN(x)$ or $x \in SN(y)$ then the elements x or y are called incident to each other.

According to the above definition, the incidence relation is symmetric and reflexive. The notion of incidence elements is particularly similar to the adjacency relations introduced by Rosenfeld in his graph theoretic approach (Rosenfeld & Kak, 1976)

Incident path and connectedness

Let T be a subset of the space S. A sequence $(a_i \in T)^k_{i=1}$ in which subsequent elements are incident to each other is called an incidence path in T from a_1 to a_k

Incident elements are directly connected. A subset T of a space S is connected if and only if for any two elements of T there exists an incidence path containing these two elements which completely lie in T

Axiom 1 and 2 describe the connectedness properties in a locally finite space. In order to describe axioms that faithfully interpret the boundary properties that following definitions must precede;

Frontier

The topological boundary (or frontier) of a nonempty subset T of the space S is the set of all elements $x \in S$ such that each neighborhood of x contains elements of both T and its complement $S-T$. The frontier of T is denoted by $Fr(T,S)$

A pair (x,y) of elements of the frontier $Fr(T,S)$ of a subset $T \subset S$ are opponents of each other if $x \in SN(y), y \in SN(x)$ then one of them belongs to T and the other to T^c. According to the definition, the frontier of T is the same as the frontier of its complement $S-T$

The frontier $Fr(T,S)$ of a subset $T \subset S$ is called thick if it contains at least one pair of opponents. Otherwise, it is called thin

Axiom 3: The frontier $Fr(T,S)$ of any subset $T \subset S$ is thin

An important property of the frontier is, loosely speaking, that it must have no gaps. More precisely, this means that the frontier of a frontier F is the same as F

Axiom 4: The frontier of $Fr(T,S)$ is the same as $Fr(T,S)$, that is, $Fr(Fr(T,S)) = Fr(T,S)$

Khalimsky topology

The rigorous definition of the Khalimsky topology was given in Khalismsky et al., (1990). This definition is summarized as below;

Let X be a connected topological space such that if Y is a three point subset of X then there exists $y \in Y$ such that Y meets two connected components of $X - \{y\}$ i.e. for any three points, one of them 'separates' the other two. A topology on the digital plane is conveniently defined by determining a topology on the integers. This is achieved by means of a minimal or smallest neighborhood $SN(x)$ for each element x. Thus,

$$SN(k) = \begin{cases} k, & k \text{ is even} \\ \{k-1, k, k+1\}, & k \text{ is odd} \end{cases}$$

This space can be regarded as a digital line since it satisfies the condition of being a connected ordered topological space. The topology on the integers and the associated product topologies are called the Khalimsky Topologies.

Digital Jordan Curve theorem

To formulate a digital Jordan curve theorem requires that set S and its complement S^c to have different ideas of adjacency, usually S is defined as 4-connected and S^c is 8-connected. Hence,

The complement of a curve γ has exactly two components, the inside and the outside (the one that meets the background) of γ. Moreover, every point of γ is 4-adjacent to both of these components, (Kovalevsky, 2006)

An approach to digital topology that is considered to be meaningfully valid must be in accordance to classical topology with regard to connectedness and validity of the Jordan curve theorem. The Jordan curve theorem faithfully defines boundary properties of a space, (Kong, 2001)

In order to be explicitly representable in the computer, a digital space must of necessity be a locally finite space. A locally finite space is one for which every point in the space has a neighborhood that meets only finitely many elements of the cover. In other words, every point has a finite neighborhood.

Continuous functions on fuzzy digital geometry

The notion of continuous functions on digital pictures was introduced by Rosenfeld (1986) and further explored by Boxer (1994). In Nakumura, (1996), Continuous functions were extended to fuzzy digital pictures and various analogous properties to those on crisp digital pictures were shown.

Digitally continuous functions

Let $X \subset I_k$ and $Y \subset I_m$ be digital images. Let $f : X \rightarrow Y$ be a function. Then f is called a digitally continuous function at $x_0 \in X$ *iff* for every $\varepsilon \geq 1$ there exists a $\delta \geq 1$ sucht that $\forall x \in X$ and $d_k\left(x_0, x\right) \leq \delta$ implies $d_m\left(f\left(x_0\right), f\left(x\right)\right) \leq \varepsilon$, (Rosenfeld 1986).

f is digitally continuous *iff* it is digitally continuous at every $x \in X$

Several authors extended the work by Rosenfeld (1986), to the realm of fuzzy digital topology including Nakamura & Aizawa, (1991) and Nakamura (1994). This characterization is provided below.

Characterization of digitally continuous functions on fuzzy sets

Suppose $\left(P, \mu(P)\right)$ represents a point P with a fuzzy value $\mu(P) = u$ in the fuzzy subset μ, then f is called a map if $f\left(P, \mu(P)\right) = \left(Q, \delta(Q)\right)$, i.e., f maps a point P with a fuzzy value $\mu(P)$ to a point Q with a fuzzy value $\delta(Q)$, (Nakamura & Aizawa, 1991).

In a subsequent paper, Nakamura (1996) generalized the definition of continuity in the following way

Appendix F: Full Python implementation of the BFJS Algorithm

```
mu2_plus = sigmoid(I2, T2, k=0.1)


mu1_minus = 1 - mu1_plus

mu2_minus = 1 - mu2_plus


# Compute Bipolar Jaccard Similarity

numerator = np.sum(np.minimum(mu1_plus, mu2_plus)) - np.sum(np.maximum(mu1_minus, mu2_minus))

denominator = np.sum(np.maximum(mu1_plus, mu2_plus)) + np.sum(np.maximum(mu1_minus, mu2_minus))


bipolar_jaccard_similarity_sigmoid = numerator / denominator


# Print the result

print(f"Bipolar Fuzzy Jaccard Similarity: {bipolar_jaccard_similarity_sigmoid:.4f}")


# Visualization of fuzzy memberships

fig, axes = plt.subplots(2, 3, figsize=(12, 8))


# Original Matrices

axes[0, 0].imshow(I1, cmap="gray", vmin=0, vmax=255)

axes[0, 0].set_title("Image 1 (Grayscale)")

axes[0, 0].axis("off")


axes[0, 1].imshow(mu1_plus, cmap="coolwarm", vmin=0, vmax=1)

axes[0, 1].set_title("Bipolar Membership (Image 1)")

axes[0, 1].axis("off")


axes[0, 2].imshow(mu1_minus, cmap="coolwarm", vmin=0, vmax=1)

axes[0, 2].set_title("Negative Membership (Image 1)")
```

```
import numpy as np

import matplotlib.pyplot as plt


# Define the two grayscale images as 4x4 matrices

I1 = np.array([

    [50, 80, 120, 200],

    [30, 90, 130, 210],

    [60, 100, 140, 220],

    [70, 110, 150, 230]

], dtype=np.float32)


I2 = np.array([

    [55, 85, 125, 205],

    [35, 95, 135, 215],

    [65, 105, 145, 225],

    [75, 115, 155, 235]

], dtype=np.float32)


# Compute the threshold (T) as the mean intensity of each image

T1 = np.mean(I1)

T2 = np.mean(I2)


# Define the sigmoid function for bipolar fuzzy membership

def sigmoid(x, T, k=0.1):

    return 1 / (1 + np.exp(-k * (x - T)))

# Compute positive and negative membership functions

mu1_plus = sigmoid(I1, T1, k=0.1)
```

Brief Explanation of the Implementation

1. Matrix Definition
   - The two grayscale images are defined as 4×4 matrices.
2. Sigmoid-Based Bipolar Membership Computation
   - The threshold T is computed as the mean intensity of each image.
   - The sigmoid function is applied to compute fuzzy positive and negative memberships for each pixel.
3. Bipolar Fuzzy Jaccard Similarity Calculation
   - The intersection (min) and union (max) of the memberships are computed.
   - The final similarity score is calculated using the Bipolar Jaccard Similarity formula.
4. Visualization
   - The original images, bipolar fuzzy memberships, and negative fuzzy memberships are displayed using matplotlib.

Appendix G: Research Permit

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Date of Issue: **06/February/2024**

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