INTERVAL ESTIMATION FOR A TWO-PARAMETER WEIBULL DISTRIBUTION BASED ON TYPE-2 CENSORED DATA

MWELELI RAPHAEL MASILA

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DECLARATION AND RECOMMENDATION

Declaration

This thesis is my original work and has not been presented in this university or any other for the award of a degree.

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Mweleli Raphael Masila

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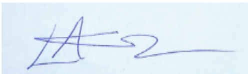
Recommendation

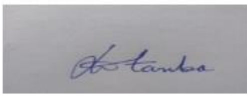
This thesis has been submitted with our approval as Egerton University supervisors

Signature…  Date…4/4/2024………..

Dr. Luke Akong'o Orawo (Ph.D.)

Mathematics Department

Egerton University

Signature…  Date…4/4/2024………..

Dr. Cox Lwaka Tamba (Ph.D.)

Mathematics Department

Egerton University

DEDICATION

This thesis is dedicated to my dear parents Joseph Mweleli and Monica Mwikali.

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ABSTRACT

In most occasions, when performing life testing experiments, the main interest is to examine the lifespan of a specimen. For example, the time an aircraft wing takes until it fails from metal fatigue, or the survival time of a patient after a kidney transplant. In practice, such life data is usually censored because one does not have sufficient resources in terms of money and time. Type-2 censoring is one of the most popular censoring schemes used in reliability and life testing experiments. Weibull distribution is the most preferred distribution to fit censored life data because it is versatile and able to take on characteristics of other types of statistical distributions based on the value of the shape parameter. The maximum likelihood method is applicable for obtaining ML estimates (MLEs) for parameters of the 2-parameter Weibull distribution. Once the parameter point estimates have been obtained, construction of confidence intervals and confidence regions can be performed. In previous research works, construction of approximate confidence intervals based on Wald method under type-2 censoring scheme has been done. However, these confidence intervals may not be accurate for small sample sizes. The profile-likelihood method can be used to construct approximate confidence intervals for the parameters of interest when the sample size is small. In this study, the approximate profile-likelihood confidence intervals and likelihood confidence region are constructed for parameters of the 2-parameter Weibull distribution based on small type-2 censored samples. The study employed both simulated and real data sets. Subroutines for construction of profile- likelihood intervals were developed in *R* program (version 3.5.1). Approximate profile-likelihood confidence interval results were then compared with the Wald confidence intervals using confidence lengths and coverage probabilities. Most of the coverage probability results for the parameters associated with the Wald method were relatively unstable because they were below the nominal coverage probability (0.95). On the other hand, most of the coverage probabilities associated with the profile-likelihood method were relatively close to the nominal coverage probability (0.95). The Profile-likelihood method outperformed Wald method because the confidence lengths obtained using profile-likelihood technique were narrower as compared to those associated with Wald method. Finally, Profile-likelihood interval estimates obtained in this study using small type-2 censored data can be used to make better inferences in life-testing experiments by using an effective small sample size.

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LIST OF ABBREVIATIONS AND ACRONYMS

CDF	Cumulative Density Function
CI	Confidence Interval
EM	Expectation Maximization
GLS	Generalized Least Square
MLE	Maximum Likelihood Estimate
MOM	Method of Moments
MSE	Mean Squared Error
PDF	Probability Density Function
RMSE	Root Mean Squared Error

LIST OF SYMBOLS

$E(X)$	Mean of X
$f(X)$	Probability Density Function
$F(X)$	Cumulative Density Function
$h(X)$	Hazard Function
$H(X)$	Cumulative Hazard Function
$R(X)$	Reliability Function
$R(\alpha)$	Relative Likelihood Function for α
$R(\beta)$	Relative Likelihood Function for β
$R(\alpha, \beta)$	Relative Likelihood Function for α and β
$var(X)$	Variance of X
X	Random variable

CHAPTER ONE

INTRODUCTION

1.1 Background Information

Censoring refers to a condition in which the complete information about a variable of interest is not fully available or is partially known. Censoring in life test experiments happens when observations about the time to failure of an object or individual is not completely observed. Censoring of life data is a common phenomenon in reliability studies and survival analysis (Sen, 1995). For instance, tests might be carried out to determine the lifespan of an aircraft wing prior to failure from metal fatigue. Such experiments are expensive and time consuming, and only a few units can be inspected. Due to time and financial constraints, a researcher may not be able to examine the failure time of all the units under investigation (Park & Lee, 2012; Stotvig, 2014). One is prompted to set an appropriate censoring limit after which the experiment is terminated. Censoring is commonly applied in numerous fields such as engineering, medicine, biology, and economics.

Different types of censoring exist including; right censoring, left censoring, interval censoring, type-1 censoring, type-2 censoring, among others (Lawless, 2011). Some of the common statistical distributions used to fit censored life data include Rayleigh distribution (Cho, 2014), Gamma distribution (Balakrishnan & Mitra, 2013), Gumbel distribution (Sindhu *et al.*, 2016), logistic distribution (Yenilmez, 2018), lognormal distribution (Emura & Shiu, 2014), log-logistic distribution (Guure *et al.*, 2014), exponential distribution (Khoolenjani & Shahsanaie, 2016), normal distribution (Kim, 2014), Weibull distribution (Ferreira & Silva, 2017; Pradhan & Kundu, 2014), and Mixed Weibull distribution (Ateya & Alharthi, 2014).

From the aforementioned life distributions, Weibull distribution is the most preferred distribution because it can take characteristics of other life distributions based on the value of shape parameter β (Nelson, 1990). Although, there are different forms of Weibull distributions, 2-parameter Weibull distribution and 3-parameter Weibull distribution are the most commonly used forms in fitting censored life data. The 2-parameter Weibull distribution has the shape parameter β and the scale parameter α. The 3-parameter Weibull distribution has the shape parameter β, scale parameter α and the location parameter Υ .

Estimation of parameters for a 2-parameter Weibull distribution or three-parameter Weibull distribution in the presence of censored data can be done using various techniques. These methods include maximum likelihood estimation (MLE) (Karakoca & Pekgor, 2019), method of moments (MOM) (Sirvanci & Yang, 1984), expectation maximization (EM) algorithm (Ferreira & Silva, 2017), and generalized least square (GLS) (Engeman & Keefe, 1982).

After parameter estimation, construction of approximate confidence intervals or approximate confidence regions for the parameters of interest may be obtained. Confidence intervals depict plausible bounds of parameter estimates. Interval estimation in a 2-parameter Weibull distribution based on large samples has been carried out in several ways; including Bayesian and Wald methods (Jeng & Meeker, 2000). In previous research works, Wald method has been employed to construct approximate confidence intervals for the parameters of this distribution under type-2 censoring scheme (Panahi, 2011). However, this method is appropriate for large samples and may give inaccurate interval estimates when the sample size is small. Hence, there is a need to try other available methods of constructing confidence intervals, especially on small samples. The profile-likelihood technique comes in handy.

The profile-likelihood method is used in situations where the focus is on a subset of parameters of a certain statistical model with the rest of the parameters considered as nuisance. Under this method, the multiparameter likelihood function is maximized over the nuisance parameters to express them as functions of parameters of interest and substituting them in the likelihood function. To facilitate the construction of the profile-likelihood confidence interval for the parameter of interest, the concept of relative profile likelihood is employed. The relative profile likelihood function of the parameters of interest is defined as the ratio of the profile likelihood function to its maximum value. Suppose that the relative profile likelihood function is for a single parameter of interest. Then on the basis of the graph of the relative profile likelihood function, the approximate confidence interval can be constructed for the parameter of interest. This interval estimate is called profile likelihood confidence interval and is defined as follows. For a fixed p, such that $0 < p < 1$, the set values for the parameter of interest whose relative profile likelihood function is greater than or equal to p is called a $100p\%$ profile-likelihood confidence interval (Kabfleisch, 2012).

In this study, 2-parameter Weibull distribution was used to fit type-2 censored data. MLE method was used to find the ML estimates for the two-parameter Weibull distribution. Once parameter estimates were obtained, approximate confidence intervals and confidence regions for the two parameters of the Weibull distribution were computed using profile- likelihood method. The study employed small samples to construct profile -confidence intervals and likelihood confidence regions. These confidence intervals were then compared with those of the Wald method using confidence lengths and coverage probabilities.

1.2 Statement of the Problem

A major problem associated with censoring is that it renders the affected dataset incomplete. Failure to select an appropriate life distribution that could be used to fit the censored data may yield invalid statistical results. Although various life distributions for fitting life censored data exist, Weibull distribution proves more efficient because it can take on attributes of some other life distributions based on the value of the shape parameter β. The parameter estimates of the Weibull distribution may be obtained using the MLE technique. It is important to construct confidence intervals and confidence regions of these parameters to enable us make irrefutable statistical inferences. Wald method has been employed to construct approximate confidence intervals or approximate confidence regions under type-2 censoring scheme. However, this method is appropriate for large samples and may give inaccurate interval estimates when the sample size is small. Hence, there is a need to explore other methods of constructing confidence intervals, especially on small samples. This study considered profile- likelihood method in constructing confidence intervals and confidence regions for the parameters of the Weibull distribution based on small type-2 censored samples.

1.3 Objectives

1.3.1 General Objective

To perform interval estimation on the parameters of the 2-parameter Weibull distribution based on type-2-censoring scheme.

1.3.2 Specific Objectives

i. To obtain profile-likelihood confidence intervals for the two parameters of the Weibull distribution.

ii. To determine the efficiency of profile-likelihood confidence intervals and Wald confidence intervals for the two-parameter Weibull distribution using confidence lengths and coverage probabilities.

iii. To construct approximate likelihood confidence regions for the two-parameter Weibull distribution.

1.4 Assumptions

i. n identical items will be subjected to a life test.

ii. The failure times of the items are independent.

1.5 Justification of the Study

Censoring is an important statistical aspect in many real-world situations. For instance, some industrial products have considerably long-life spans. This property makes it infeasible to track the failure time of all the products. Accelerated testing may not yield the desired results. A researcher may, therefore, decide to terminate the experiment once a specific number of failed products has been recorded. In such situations, type-2 censoring has to be employed. It is important to account for all the censored information accurately. Being a flexible model, 2-parameter Weibull distribution is one of the most efficient distributions for fitting censored life data. Although there are various techniques that can be used to obtain the estimates of parameters for the Weibull distribution, the MLE technique is particularly preferred because it is relatively easy to compute. Additionally, ML estimates with no closed forms can still be obtained by employing numerical methods such as the Newton- Raphson technique. Due to limited time and financial constraints, a researcher would prefer a small sample over a large one for analysis. However, the analyst would need to employ an appropriate statistical technique in constructing the confidence intervals and regions to make plausible statistical inferences about the model parameters. Wald method has been considered in constructing confidence intervals for 2-parameter Weibull distribution. The technique yields inaccurate results, especially on small samples. There is a need to consider other methods of constructing confidence intervals for parameter estimates of this distribution. This study focused on the profile- likelihood technique of constructing confidence intervals. The efficiency of this technique in carrying out interval estimation for 2-parameter Weibull distribution based on type-2 censored data was compared with the Wald technique.

CHAPTER TWO

LITERATURE REVIEW

2.1 Overview of Censoring

Censoring is a common phenomenon in reliability studies and survival analysis because most of the time when performing life test experiments, one does not have adequate resources in terms of money and time. In such cases, shortening the project total lifetime may be allowed by setting an appropriate censoring limit. We have different types of censoring including right censoring, left censoring, interval censoring, type-l censoring and type-2 censoring. Type-1 and type-2 censoring schemes are the two most popular censoring schemes used in reliability and life testing experiments (Panahi, 2011). This is because most of the life test experiments especially in the field of engineering and medicine require one to choose a combination of time and sample size that will minimize costs.

2.1.1 Right Censoring

Right censoring occurs if the failure time is known to be higher than a certain value but is not known by how much (Stotvig, 2014). For example, suppose a reliability engineer subjects ten fan belts to a life test. Four fan belts fail in 70 hours, 84 hours, 98 hours, 109 hours and 135 hours. When the engineer terminates the experiment at 150 hours, he notices that the remaining six fan belts are still operating. These last six fan belts are right censored at 150 hours.

2.1.2 Left Censoring

Left censoring occurs if the failure time is lower than a certain value but is not known by how much. Under this censoring scheme, failures occur before a particular time (Stotvig, 2014). For example, twenty capacitors are subjected to a life test at high voltage levels to accelerate their failure times. Afterwards, an electrical engineer examines the capacitors after every ten hours to find out how many have failed. The specific time at which every capacitor fails is not considered important. At the first inspection, three are recorded to have failed. The failure times for these three capacitors are left censored.

2.1.3 Interval Censoring

Interval censoring arises when an event occurs over a duration of time, but the exact time of the event is unknown. Consider an example where n identical items are subjected to a life test. Let $t_1, \ldots, t_n$ denote lifetime of these items. Let the unknown failure time to be denoted by t_p such that it falls on the interval $t_1 \leq t_p \leq t_n$ (Pradhan & Kundu, 2014). Such a scheme is referred to as interval censoring. In this type of censoring, the researcher is not certain as to when units will actually break down.

2.1.4 Type -1 Censoring

Type-1 censoring occurs if a set number of identical items are subjected to a life test and the experiment is stopped at a predetermined time; the remaining items are then right censored. Under this type of censoring, the total time of study is fixed while the number of censored units is random (Stotvig, 2014).

2.1.5 Type-2 Censoring

Type-2 censoring arises when a set of identical items are subjected to a life test, and the experiment is stopped when a predetermined number of items are observed to have failed, and after that, the remaining items are right censored (Panahi, 2011). Here, the total number of censored units is fixed while the time of study is random

2.2 Censoring of Life Data

The term life data refers to measurements of the lifespan of an item or times-to-failure of the item. The most common statistical distributions used to fit censored life data include Rayleigh distribution, Gamma distribution, logistic distribution, lognormal distribution, exponential distribution, normal distribution, Weibull distribution, log-logistic distribution, Gumbel distribution, and mixed Weibull distribution.

Cho *et al.* (2014) performed an estimation of the entropy for a Rayleigh distribution based on doubly-generalized type-2 hybrid censored samples. Through a simulation study, they obtained entropy estimators using the MLE, approximate MLE and Bayesian technique and compared the efficiency of the three techniques using root mean squared error (RMSE). Bayesian technique was found to perform better than the other two techniques because estimates under Bayesian approach yielded the smallest RMSE.

Balakrishnan and Mitra (2013) applied Gamma distribution to fit left truncated and right censored data. They used the expectation maximization (EM) algorithm and Newton-Raphson method to estimate the scale and shape parameters of the gamma distribution. They further computed approximate confidence intervals for the scale and shape parameter based on the EM algorithm and Newton-Raphson method and compared the efficiency of the two techniques in terms of coverage probabilities. They found out that the coverage probabilities corresponding to the two methods were relatively close.

Sindhu *et al. (*2016) considered left censored data from a Gumbel distribution and used Bayesian approach to find the parameter estimates of the distribution. The simulation study performed revealed that the Bayesian credible intervals assuming inverse levy prior were much narrower than the credible intervals assuming informative and non-informative priors.

Yenilmez *et al.* (2018) used a censored regression model based on generalized logistic distribution to solve the problem of Tobit estimators which are inefficient in the context of non-normal errors. To solve this problem, they considered partially adaptive estimators based on generalized logistic distribution (PAEGLD) which is more flexible than normal distribution and is able to accommodate kurtosis. Based on results from the simulation study, they found that PAEGLD presented very little loss in terms of efficiency as compared to Tobit estimator for a normal error distribution. They also found PAEGLD to be more useful in the context of small samples and more robust to the underlying distributional assumptions than Tobit.

Emura and Shiu (2014) applied lognormal distribution to fit left-truncated and right-censored data in the analysis of life-time of electric transformers. They further did a comparison of the EM algorithm with the Newton-Raphson algorithm in terms of convergence performance. Through a simulation study, they found out that the Newton-Raphson algorithm performed better than EM algorithm. This is because the number of iterations in the Newton-Raphson algorithm was smaller than in the EM algorithm when computing ML estimates of the parameters of the lognormal distribution.

Khoolenjani and Shashsanaie (2016) used exponential distribution to fit fuzzy data from the type-2 censoring scheme. They addressed the Bayesian inference for the parameter of the exponential distribution based on type-2 censoring scheme. Based on the simulation study results, they found out that using Bayes informative prior yields improved estimates in terms of mean squared error (MSE).

Kim (2014) applied normal distribution to fit type-2 censored data. The obtained ML estimates for parameters of the normal distribution were then compared with Gupta (1952) linear estimators. Through a simulation study, the approximate MLEs were found to be more efficient and to perform better than Gupta linear estimators.

Pradhan and Kundu (2014) performed both classical and Bayesian analysis of interval censored data that followed a 2-parameter Weibull distribution. Through a simulation study, they found out that ML estimates did not have explicit forms and therefore proposed the EM algorithm for computation of the ML estimates. They also found out that the Bayes estimates did not exist in explicit form under the squared error loss function. They, therefore, proposed an importance sampling technique to be deployed to obtain Bayes estimates when the scale and shape parameters have independent gamma priors.

Ferreira and Silva (2017) applied Weibull distribution to fit the right censored data. They analyzed data obtained from the historical record of five centrifugal pump failures of a petrochemical company. Due to difficulty of the analyzed data, they found out that maximum likelihood equation showed no analytical solution and they proposed expectation maximization (EM) algorithm. Ateya and Alharthi (2014) obtained ML estimates of the parameters of a finite mixture of modified Weibull distributions based on type-1 and type-2 censored samples using EM algorithm. Through a simulation study, they found out that modified mixture Weibull model fitted the data better than mixture Weibull model.

Guure (2015) used log-logistic to fit right censored data from survival of patients with cervical cancer. Bayesian credible interval was then computed using the squared error loss function and LINEX loss function for the parameters of the model. For the scale parameter, LINEX loss function with a positive loss parameter had narrower credible interval as compared to squared error loss function. For the scale parameter, Bayesian credible interval with squared loss error were narrower than the Bayes using LINEX loss function.

Wahed *et al.* (2009) investigated the usefulness of the Beta-Weibull distribution in modelling censored survival data based on biomedical studies. In their study, they fitted the Beta-Weibull distribution and 2-parameter Weibull distribution to a dataset from a breast cancer study conducted by Surgical Adjuvant Breast and Bowel Project Study (NSABP). The 2-parameter Weibull distribution exhibited monotone hazard shapes but did not reflect the pattern of the hazard shape in the observed dataset. Beta-Weibull distribution on the other hand was able to accurately capture the observed hazard pattern.

From the aforementioned life distributions, Weibull distribution is the most preferred distribution because it is versatile and can take on the characteristics of other types of statistical distributions based on the value of the shape parameter β (Nelson, 1990).

2.3 Weibull Distribution

The Weibull distribution is a lifetime distribution that was first introduced by Swedish mathematician and engineer, Professor Wallodi Weibull (1951). Since its introduction, Weibull has been widely studied and applied in many fields such as engineering, medicine, finance, chemistry, material science, safety and testing, among others. Weibull distribution is one of the most efficient life distributions in reliability studies and life testing problems such as time to failure or checking the durability of a product (Ahmed, 2013). There are different types of

Weibull distribution, but the most commonly used forms are the 2-parameter Weibull distribution and the 3-parameter Weibull distribution.

2.3.1 Two-Parameter Weibull Distribution

A continuous random variable X is said to have a 2-parameter Weibull distribution if its probability density function is given by equation (2.1):

$$f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, x > 0 \; ; x > 0, \alpha > 0, \beta > 0 \qquad (2.1)$$

where α and β are the scale and shape parameters, respectively. This form of the Weibull distribution is known as the 2-parameter Weibull distribution.

The mean for the 2-parameter Weibull distribution is given by

$$E(X) = \alpha^{-\frac{1}{\beta}}\Gamma(1 + \frac{1}{\beta}) \qquad (2.2)$$

The variance is expressed as

$$var(X) = \alpha^{-\frac{2}{\beta}}\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\alpha^{-\frac{1}{\beta}}\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \qquad (2.3)$$

The CDF for the 2-parameter Weibull distribution is given by

$$F(x) = P(X \leq x) = 1 - e^{-\alpha x^{\beta}}$$

$$= 1 - e^{H(x)} \qquad (2.4)$$

where $H(x)$ is the cumulative hazard function and $h(u)$ is the hazard function.

$$H(x) = \int_0^x h(u)\, du = -\alpha x^{\beta} \qquad (2.5)$$

2.3.2 Reliability

Reliability of an item refers to the probability of that item surviving beyond some specific time, say x (Newby, 1979). Let X denote time to failure or failure time of the item. Then X is continuous nonnegative random variable with CDF $F(x)$. The reliability, denoted by $R(x)$, at time x of an item is given by

$$R(x) = P(X > x) = 1 - F(x)$$

$$= e^{H(x)} \qquad (2.6)$$

2.3.3 Hazard Function

Hazard function is the probability of failure in an infinitesimally small period of time between x and x+Δx given that an item has survived until time x (Bain and Engelhardt, 1992).

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{R(x)}$$

$$= \alpha\beta x^{\beta-1} \tag{2.7}$$

Alternatively;

$$h(x) = \frac{-d[log\, R(x)]}{dx} \tag{2.8}$$

A function $h(x)$ is a hazard function if and only if it satisfies the following properties as presented by Bain and Engelhardt (1992):

$$h(x) \geq 0 \text{ for all } x \tag{2.9}$$

$$\int_0^\infty h(x)\, dx = \infty \tag{2.9.1}$$

When $\beta < 1$, it implies that the failure rate or hazard rate decreases over time. This situation occurs if we have defective products failing early and the hazard rate decreasing over time as defective products are removed.

If the value of $\beta=1$, then it means that the failure rate is constant over time. This might propound that random external factors are causing mortality or failure and hence, Weibull distribution reduces to an exponential distribution.

A value of $\beta > 1$ implies that the failure rate increases over time. This is common in electronic gadgets because most of them are likely to fail as they age.

2.4 Parameter Estimation

A variety of statistical methods for estimating model parameters under various censoring schemes exist in literature. They include maximum likelihood estimation (MLE), expectation maximization (EM) algorithm, and method of moment (MOM).

2.4.1 Method of Moment (MOM)

Moment estimators are often used because they are relatively easy to compute (Kantar & Senoglu, 2008). Sirvanci and Yang (1984), applied MOM to estimate parameters of the

Weibull distribution under the type-1 censoring scheme. To use the MOM, one has to derive the first and the second population moments and equate them to corresponding sample moments. According to George (2014), if the underlying statistical distribution is a two-parameter Weibull distribution, then its 1st and 2nd moments are respectively

$$\mu_1' = E(X) = \alpha^{-\frac{1}{\beta}}\Gamma\left(1+\frac{1}{\beta}\right) \tag{2.9.2}$$

$$\mu_2' = E(X^2) = \alpha^{-\frac{2}{\beta}}\Gamma\left(1+\frac{2}{\beta}\right) \tag{2.9.3}$$

The sample moments are respectively

$$M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{2.9.4}$$

$$M_2' = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \tag{2.9.5}$$

Finally, the above equations are solved simultaneously to obtain moment estimates.

2.4.2 Method of Maximum Likelihood Estimation (MLE)

The maximum likelihood estimation (MLE) is a popular method for estimating parameters of a statistical model. For a given dataset and underlying statistical model, ML method provides estimates of the model parameters that render the observed data maximal probability of occurrence. According to Cohen (1965), Harter and Moore (1965), Ross (1994), and Cole *et al.* (2013), ML method is popular because it is relatively easy to compute and possesses adorable large sample properties in instances where the model has been correctly specified.

The ML method maximizes the log-likelihood function to get parameter estimates of the underlying statistical model.

In reliability engineering and survival analysis, Weibull distribution is preferred to other statistical life distributions to fit life data as earlier discussed. Life dataset can be complete or censored. In the context of complete data, the ML method is an efficient method of estimating parameters of a Weibull distribution (George, 2014; Lawless, 1982). Let $X_1, \ldots, X_n$ to be a random sample of size n from 2 –parameter Weibull (α, β) distribution. The log-likelihood function is given by

$$\log L(\alpha,\beta) = n \log \alpha + n \log \beta + (\beta - 1)\sum_{i=1}^{n} \log X_i - \alpha \sum_{i=1}^{n} X_i^{\beta} \tag{2.9.6}$$

Equation (2.9.6) is differentiated partially with respect to α and β in order to obtain the respective MLEs.

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} x_i^{\beta}} \tag{2.9.7}$$

The parameter β does not have a closed form solution. To obtain its ML estimate, numerical methods such as the Newton -Raphson iterative procedure are employed.

The type-2 censoring scheme is one of the most well-known censoring schemes used in reliability and life testing experiments (Joarder *et al.*, 2011; Panahi, 2011). Under this scheme, the method of MLE can be used to estimate parameters of the Weibull distribution and is outlined as follows.

Afterwards, the parameterized distribution for the censored dataset can be used to estimate consequential life characteristics of the item such as reliability or probability of failure at a particular time, the mean life, and the failure rate.

2.4.3 Expectation Maximization (EM) Method

The EM algorithm is a parameter estimation technique introduced by Dempster *et al.* (1977). It is an iterative process used to compute the ML estimators in instances with incomplete data or when the value of censored data is partially known. The method has two steps: the expectation (E) step and the maximization (M) step. The E step generates an expectation of log-likelihood function calculated using current parameter estimates. The M step calculates the parameter values maximizing the expected log-likelihood function in the E-step. Ferreira and Silva (2017) performed the estimation of Weibull distribution parameters with right censored data using the EM algorithm. Vishwakarma *et al.* (2018) described the estimation of Weibull distribution parameters based on fuzzy data using the EM algorithm. Elmahydy and Aboutahoun (2012) used the EM algorithm to estimate parameters of finite Weibull mixture distribution for reliability modelling. Nandi *et al.* (2010) comprehensively discussed the estimation of the Marshall-Olkin Bivariate Weibull distribution parameters in the presence of random censoring using EM algorithm.

2.5 Interval Estimation

In statistics, interval estimation refers to the use of sample data to compute an interval of probable values of an unknown population parameter. In contrast, point estimation uses a single value to make inference about an unknown population parameter of interest (Neyman, 1937). Interval estimates may be used as a substitute or as an appendage to the point estimates to make valid statistical inferences about the unknown population parameter. Occasionally, if we want to compute interval estimates, we need an assurance that the interval will contain a plausible value of the parameter of interest with a given probability. Such intervals are known as confidence intervals.

In practice, approximate confidence regions or confidence intervals for large samples are determined by applying aspects of the underlying statistical distribution of the data and by employing the concept of normal approximations. In the context of small samples, confidence intervals or confidence regions based on likelihood ratio are computed to obtain plausible interval estimates to make irrefutable statistical inference about the unknown population parameter of interest. Weibull distribution is one of the most preferred life distributions to fit complete or censored life data. After obtaining the point estimates of the parameters of a Weibull distribution using the aforementioned estimation methods, confidence intervals of the shape parameter β and scale parameter α may be computed. These interval estimates indicate the uncertainty of the point estimates and hence serve as measures of accuracy of point estimation of the population parameters based on the observed data. In literature, different methods of constructing approximate confidence regions and intervals for these parameters exist, namely, Wald and profile- likelihood methods.

2.5.1 Wald Method

Wald statistics were developed in mid-20th century by Abraham Wald. In application, confidence intervals hinged on normal approximation theory of ML estimates are common (Jeng, 1998). The Wald statistics is a first order asymptotic statistic derived from a function of the Fisher's information matrix (Hewage, 2018).

2.5.2 Profile-Likelihood Method

In many instances, the likelihood is a function of multiple parameters, but a statistician maybe interested in a subset of the parameters, with the others considered as nuisance parameters. To achieve this goal, the profile-likelihood method is employed to reduce the likelihood to a function of the few selected parameters of interest by treating the rest as nuisance parameters. The likelihood function is then maximized over the nuisance parameters to obtain the profile likelihood of the parameters of interest. Consider a lifetime dataset with the underlying distribution being the 2-parameter Weibull distribution. The relative likelihood function of α and β , denoted by $R(\alpha,\beta)$, is given by

$$R(\alpha,\beta) = \frac{L(\alpha,\beta)}{L(\widehat{\alpha},\widehat{\beta})}$$

$$= \frac{\log n! - \log(n-r)! + r\log\alpha + r\log\beta - (n-r)\alpha x_r^{\beta} - \alpha\sum_{i=1}^{r} x_i^{\beta} + (\beta-1)\sum_{i=1}^{r} \log x_i}{\log n! - \log(n-r)! + r\log\widehat{\alpha} + r\log\widehat{\beta} - (n-r)\widehat{\alpha} x_r^{\widehat{\beta}} - \alpha\sum_{i=1}^{r} x_i^{\widehat{\beta}} + (\widehat{\beta}-1)\sum_{i=1}^{r} \log x_i} \quad (2.10)$$

Suppose that β is our parameter of interest and α to be the nuisance parameter. The relative profile likelihood function for β , denoted by $R_p(\beta)$, is obtained by maximizing R (α, β) over α with β fixed (Kabfleisch, 1985). That is,

$$R_p(\beta)=Max_\alpha R(\alpha,\beta)=R[\hat{\alpha}(\beta),\beta] \quad (2.11)$$

Similarly, if we let α be our parameter of interest and β to be the nuisance parameter. The relative profile likelihood function for α is obtained by maximizing R (α, β) over β with α fixed.

$$R_p(\alpha)=Max_\beta\, R(\alpha,\beta)=R\left[\hat{\beta}(\alpha),\alpha\right] \quad (2.12)$$

Note that the relative profile-likelihood function of a parameter takes a value between zero and one.

2.6 Likelihood Regions

Let $R(\alpha,\beta)$ be the joint relative likelihood function defined in equation (2.10) For fixed $p\in(0,1)$, the 100p% likelihood region is the set of parameter values (α,β) such that $R(\alpha,\beta)>p$. The curve $R(\alpha,\beta)=p$ which forms the boundary of this region is called the $100p\%$ likelihood contour. They are used to assess the accuracy of both the maximum likelihood estimates and interval estimates of α and β. The graph of the joint relative likelihood function can be reproduced by reporting a few representative likelihood contours and its symmetry relative to the point $(\hat{\alpha},\hat{\beta})$ can be visualized.

The 100p% likelihood regions can be used as approximate confidence regions for (α,β) .whole coverage probabilities are obtained using the approximate chi-square distribution of the likelihood ratio statistic $-2logR(\alpha,\beta)$ for testing the hypothesis$(\alpha,\beta)=(\alpha_0,\beta_0)$. The likelihood ratio statistic has approximate chi-square distribution with 2 degrees of freedom. The true value (α_0,β_0) belong to the $100p\%$ likelihood region if and only if $logR(\alpha,\beta)\geq logp$. Therefore the coverage probability of $100p\%$ likelihood region for (α,β) is

$$\begin{aligned} P(-2logR(\alpha,\beta)\leq -2logp) &\approx P\left(\chi^2_{(2)}\leq -2logp\right) \\ &= 1-e^{logp}=1-p. \end{aligned} \quad (2.13)$$

It follows that the $100p\%$ likelihood region for (α,β) is an approximate$100(1-p)\%$ confidence region (Kabfleisch, 1985).

CHAPTER THREE

MATERIALS AND METHODS

3.1 The Probability Distribution.

This study analyzed a type-2 censored data generated by subjecting n identical items to a life test, and the test was terminated at the time of r^{th} $(r < n)$ unit failure. The failure times of first r items were used for analysis while those of the remaining $(n - r)$ items were right censored. The failure times for the items were ordered as $X_1, \dots X_r$. The 2-parameter Weibull distribution given by equation (2.1) was used to fit the type-2 censored data. Censoring is necessary in such experiments since it may take a very long time for all items to fail.

3.2 The Maximum Likelihood Estimation (MLE) Method.

Maximum likelihood estimates of the parameters of the 2-parameter Weibull distribution were obtained using Panahi (2011) procedure.

Let $X_1 < X_2 < \dots < X_r$ be a type-2 censored sample of size r acquired from a life test on n items whose lifetimes have Weibull. The likelihood function can be written as

$$L(\alpha, \beta) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f(x_i)\right]\left[1 - F(x_r)\right]^{n-r} \quad (3.1)$$

Specifically;

$$L(\alpha, \beta) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} \alpha\beta x_i^{\beta-1} e^{-\alpha x_i^{\beta}}\right]\left[1 - \left(1 - e^{-\alpha x_r^{\beta}}\right)\right]^{n-r}$$

$$= \frac{n!}{(n-r)!}(\alpha\beta)^r \left(\prod_{i=1}^{r} e^{-\alpha x_i^{\beta}}\right)\left(\prod_{i=1}^{r} x_i^{\beta-1}\right)\left[e^{-\alpha x_r^{\beta}}\right]^{n-r} \quad (3.2)$$

Then, the log-likelihood function is written a

$$\ell(\alpha, \beta) = log\, n! - log(n-r)! + r\, log\, \alpha + r\, log\, \beta - (n-r)\alpha x_r^{\beta} - \alpha \sum_{i=1}^{r} x_i^{\beta} + (\beta - 1)\sum_{i=1}^{r} log\, x_i \quad (3.3)$$

The MLE of α and β are denoted as $\hat{\alpha}$ and $\hat{\beta}$. They can be obtained as solutions of

$$\frac{\partial \ell}{\partial \alpha} = \frac{r}{\alpha} - (n-r)x_r^{\beta} - \sum_{i=1}^{r} x_i^{\beta} = 0 \quad (3.4)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{r}{\beta} - (n-r)\alpha x_r^{\beta}\, log\, x_r - \alpha \sum_{i=1}^{r} x_i^{\beta}\, log\, x_i + \sum_{i=1}^{r} log\, x_i = 0 \quad (3.5)$$

From equation (3.4), we obtain the MLE of α as a function of β.

$$\hat{\alpha} = \frac{r}{(n-r)x_r^{\beta} + \sum_{i=1}^{r} x_i^{\beta}} \quad (3.6)$$

$\hat{\beta}$ can be obtained as the solution of

$$\left(\frac{r(n-r)}{(n-r)x_r^{\beta}+\sum_{i=1}^{r}x_i^{\beta}}\right)x_r^{\beta}\, log\, x_r + \left(\frac{r}{(n-r)x_r^{\beta}+\sum_{i=1}^{r}x_i^{\beta}}\right)\sum_{i=1}^{r}x_i^{\beta}\, log\, x_i - \frac{r}{\beta} - \sum_{i=1}^{r} log\, x_i = 0 \quad (3.7)$$

To solve equation (3.7), we employ a Newton-Raphson iterative procedure.

Therefore , $\hat{\beta}$ can be obtained as a solution of the equation of the form

$$\hat{\beta}_{i+1}=\hat{\beta}_i+\frac{\left\{\frac{r}{\hat{\beta}_i}+S_{1f}-S_{3f}-\frac{r}{S_{2f+}S_{2s}}S_{3f}+S_{3s}\right\}}{\left\{\frac{r}{\hat{\beta}}+S_{4s}-\frac{r}{(S_{2f}+S_{2s})^2}\,(S_{3f}+S_{3s})^2+\frac{r}{S_{2f}+S_{2s}}\,(S_{4f}+S_{4s})\right\}} \quad (3.8)$$

where

S_{1f}=$\sum_{i=1}^{r} lnxi$

S_{2f}=$\sum_{xi}\beta$

S_{2s}=(n-r)x_r^{β}

S_{3f}=$\sum_{i=0}^{r} xi^{\beta}\, lnxi$

S_{3s}=(n-r)xr^{β}lnxr

S_{4f}=$\sum_{i=0}^{r} xi^{\beta}\,(lnxi)^2$

S_{4s}=(n-r)$xr^{\beta}(lnxr)^2$

Once $\hat{\beta}$ has been obtained, $\hat{\alpha}$ is computed as follows:

$$\hat{\alpha} = \frac{r}{S_{2f+}S_{2s}}$$

3.3 Interval Estimation

The main purpose of the study was to construct approximate confidence intervals for the parameters of the 2-parameter Weibull distribution using the Wald technique and profile-likelihood method. To construct Wald confidence intervals for the parameters of 2-parameter Weibull distribution under type-2 censoring scheme, the following steps described in Panahi (2011) are used.

Step 1: MLE of α and β are obtained as described in section 3.2.

Step 2: Computation of the observed Fisher information matrix for the MLE's from equations (3.6) and (3.7),

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{r}{\alpha^2} \quad (3.9)$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{r}{\beta^2} - (n-r)\alpha x_r^{\beta}(log\, x_r)^2 - \alpha\sum_{i=1}^{r}x_i^{\beta}\,(log\, x_i)^2 \quad (3.10)$$

and

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \frac{\partial^2 \ell}{\partial \beta \partial \alpha} = -(n-r)x_r^{\beta}\, log\, x_r - \sum_{i=1}^{r} x_i^{\beta}\, log\, x_i \tag{3.11}$$

Equations (3.9), (3.10), and (3.11) facilitate the computation of the observed Fisher information matrix denoted by

$$I(\hat{\alpha}, \hat{\beta}) = \begin{bmatrix} \frac{-\partial^2 \ell}{\partial \beta^2} & \frac{-\partial^2 \ell}{\partial \beta \partial \alpha} \\ \frac{-\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{-\partial^2 \ell}{\partial \alpha^2} \end{bmatrix}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} \tag{3.12}$$

The observed Fisher information matrix will aid in the construction of approximate confidence intervals for the parameters based on the limiting normal distribution.

Step 3: Obtaining the inverse of the observed Fisher information matrix in order to find a local estimate of the asymptotic variance-covariance matrix of the MLE. The inverse of the Fisher information matrix is then obtained as,

$$[I(\hat{\alpha}, \hat{\beta})]^{-1} = \begin{bmatrix} \hat{\sigma}^2(\hat{\beta}) & \hat{\sigma}(\hat{\beta}, \hat{\alpha}) \\ \hat{\sigma}(\hat{\alpha}, \hat{\beta}) & \hat{\sigma}^2(\hat{\alpha}) \end{bmatrix} \tag{3.13}$$

Step 4: In accordance with the asymptotic theory of MLE, the sampling distribution of $\frac{(\hat{\beta}-\beta)}{\sqrt{\hat{\sigma}^2(\hat{\beta})}}$ can be approximated by a standard normal distribution. A two -sided $100(1-\psi)\%$ normal approximate confidence interval for β is constructed as

$$\hat{\beta} \pm Z_{1-\frac{\psi}{2}} \sqrt{\hat{\sigma}^2(\hat{\beta})} \tag{3.14}$$

Analogously, the two-sided $100(1-\psi)\%$ normal approximate confidence interval for α is constructed as

$$\hat{\alpha} \pm Z_{1-\frac{\psi}{2}} \sqrt{\hat{\sigma}^2(\hat{\alpha})} \tag{3.15}$$

The $100p\%$ profile-likelihood interval for the parameter β is the set of all values for which $R_p(\beta) \geq p$. The $100p\%$ profile-likelihood interval for the parameter α is similarly defined. For a fixed $\psi \in (0,1)$, $100p\%$ profile-likelihood interval is an approximate $100(1-\psi)\%$ confidence interval for a given parameter with $p = e^{-\frac{z_{\frac{\psi}{2}}}{2}}$, where $z_{\frac{\psi}{2}}$ is the value of the standard normal random variable leaving an area equal to $\frac{\psi}{2}$ to the right (Kabfleisch, 1985). The value p corresponding to $\psi = 0.05$ is obtained as $p = e^{-\frac{1.96^2}{2}} = 0.147$. Therefore, a 95% confidence interval is approximated by a 14.7% profile-likelihood interval. The approximate confidence interval is called profile likelhood confidence interval.

These two types of confidence intervals were compared using interval length and coverage probability on the basis of simulated and real datasets. It is important to note that exact confidence intervals are not attainable for the parameters of the Weibull distribution. Other methods of interval estimation exist, such as the Bootstrap Method and Bayesian method, but this study was limited to the above mentioned methods.

3.4 Source of Data

The study used a secondary Real type-2 censored life data in Nelson (1982, pg. 509) to construct approximate confidence intervals for parameters of a 2-parameter Weibull distribution using the Wald and profile-likelihood methods. The data represents the lifetimes in hours of twelve appliance cords subjected to three flex tests and each test was terminated after the failure of the first nine appliance cords. The study analyzed the data from the second flex test which is given as follows: 57.5, 77.8, 88.0, 98.4, 102.1, 105.3, 139.3, 143.9, 148.0.

Since the empirical hazard function exhibits an increasing trend, the 2-parameter Weibull distribution can be used for analyzing the above real type-2 censored life data as described by Kundu and Raqab (2012).

3.5 Data Simulation

Simulation is defined as an imitation of a real system or process and is used mostly in scientific modelling of natural and human systems. This methodological approach is outstanding because many processes encountered in research applications are too sophisticated to be analyzed mathematically. In statistics, simulation is usually used to provide researchers with an appealing practical feedback when modelling real life data systems under theoretical setting (Kariuki *et al.*, 2014). For example, life-testing experiments especially in the field of engineering are expensive and time consuming. Simulation therefore allows researchers to determine the efficiency of the life-testing experiments before they are performed. In this study the algorithm by Newby (1979) was used to simulate type-2 censored data for the 2-parameter Weibull distribution. The algorithm consists of the following steps:

Step 1: Set $h_0 = 0$

Step 2: Generate random numbers $u_j \sim Uniform(0,1), j = 0,1, \dots, (r-1)$

Step 3: Set $h_{j+1} = h_j - \frac{log(u_j)}{n-j}, j = 0,1, \dots (r-1)$

Step 4: $X_{j+1} = H^{-1}(h_{j+1}),\ \ j = 0, 1, \dots, r - 1$.

In the above H^{-1} is the inverse of the cumulated hazard function in equation 2.5 and $X_j,\ j = 1, 2, \cdots, r$ are the simulated failure times of the first r items.

3.6 Data Analysis

The Wald confidence intervals and Profile-confidence intervals for the two parameters α and β of the Weibull distribution were constructed on the basis of a simulated and real type-2 censored samples following the steps outlined in section 3.3, respectively. In the first part of simulation study samples sizes $n = 20$ and $n = 40$ were simulated for various sets of fixed values of α, β and r and in each case the lengths of the two confidence intervals were then compared. Next, one thousand samples each of size n were generated by simulation and for each sample the two types of confidence intervals we constructed. This repeated sampling was used to compute the coverage probabilities of the two types of confidence intervals for the unknown parameters of the Weibull distribution. The coverage probability of a given type of confidence interval for a given parameter is obtained as the proportion of the one thousand confidence intervals containing the fixed value of the parameter used in simulating the data. The two confidence intervals were then compared on the basis of these coverage probabilities both for the cases of small and large samples. Subroutines to obtain the approximate Wald confidence intervals and the approximate profile-likelihood confidence intervals for the two parameters were developed in the R program (version 3.5.1). Real type-2 censored life data in Nelson (1982, pg. 509) was also analyzed. The two types of approximate confidence intervals were obtained by the two aforementioned methods and their confidence lengths compared.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, the results obtained in this study are presented. The approximate Wald confidence intervals and profile-likelihood confidence intervals for the parameters of a 2-parameter Weibull distribution under type-2 censoring scheme are constructed and displayed in tables. The first part of this chapter gives results based on simulated data and the second part analyses real life type-2 censored data given in page 509 of Nelson (1982).

4.2 The 95% Wald confidence intervals of α and β for different values of r and n

Approximate confidence intervals for the parameters α and β obtained using Wald technique were achieved by following the procedure described in section 3.3. Different fixed values of the scale parameter α, the effective sample size r, different values of the overall sample size n with value of the shape parameter β fixed as 1.5 were considered. The choice for fixing the value of the shape parameter β as 1.5 was just arbitrary and should other set of values be selected and maintained from onset, the same conclusion would still be realized.

The results for different sample sizes are presented in tables 1 and 2 below.

Table 1: Wald Confidence Intervals Based on MLE, for $\boldsymbol{n = 20}$

α	r	$[\hat{\alpha}_l, \hat{\alpha}_u]$	CL for α	$[\hat{\beta}_l, \hat{\beta}_u]$	CL for β
0.5	8	[0.19764, 1.22446]	1.02682	[0.39738, 2.08385]	1.68647
	4	[0.00000,0.82520]	0.82520	[0.0380146,2.067860]	2.02985
1.0	8	[0.22690, 1.97821]	1.75131	[0.377045, 1.74620]	1.36916
	4	[0.00000,1.39428]	1.39428	[0.066103, 1.93431]	1.86821
1.5	8	[0.22094, 2.54861]	2.32767	[0.37266, 1.60855]	1.23589
	4	[0.00000,1.86983]	1.86983	[0.084172, 1.86530]	1.78113

From the simulation results in the table above, it can be observed that for a fixed value of the scale parameter α and overall sample size n, as the effective sample size r increases, the length of the Wald confidence interval for the shape parameter β decreases. On the other hand, it can be observed that for a fixed value of the scale parameter α and overall sample size n, as the effective sample size r increases, the length of the Wald confidence interval for the scale parameter α also increases.

Table 2: Wald Confidence Intervals Based on MLE, for $\boldsymbol{n = 40}$

α	r	$[\hat{\alpha}_l, \hat{\alpha}_u]$	CL for α	$[\hat{\beta}_l, \hat{\beta}_u]$	CL for β
0.5	8	[0.043312, 1.33135]	1.28804	[0.45263, 2.01939]	1.56676
	4	[0.00000, 0.86937]	0.86937	[0.063159,2.108910]	2.04575
1.0	8	[0.00000, 2.19627]	2.19627	[0.46025, 1.81479]	1.35454
	4	[0.00000, 1.5250]	1.5250	[0.092796, 2.01836]	1.92556
1.5	8	[0.00000, 2.88451]	2.88451	[0.46492, 1.72284]	1.25792
	4	[0.00000, 2.08531]	2.08531	[0.11064, 1.96890]	1.85826

From the simulation results in table 2 above, it can be observed that for a fixed value of the scale parameter α and overall sample size n, as the effective sample size r increases, the length of the Wald confidence interval for the shape parameter β decreases. On the other hand, it can be observed that for a fixed value of the scale parameter α and overall sample size n, as the effective sample size r increases, the length of the Wald confidence interval for the scale parameter α also increases.

Moreover, it can be observed that Wald confidence interval results for the shape parameter β in table 2 are narrower than Wald confidence interval results for the shape parameter β in table 1. Also, it can be observed that the Wald confidence interval results for the scale parameter α in table 1 are narrower than Wald confidence interval results for the scale parameter α in table 2.

The approximate Wald confidence interval for the scale parameter α and the shape parameter β vary with respect to the effective sample size r because they are hinged on the asymptotic theory of MLE and from equation's (3.6) and (3.8) it can be observed that both the MLE of α and β are functions of the effective sample size r.

4.3 The 95% Profile Confidence Intervals of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for different values of $\boldsymbol{r}$ and $\boldsymbol{n}$

The 95% profile-likelihood confidence intervals for the parameters α and β given different effective sample size r and overall sample size n are also constructed following the steps described in section 3.3. The 95% profile-likelihood confidence intervals are displayed in tables 3 and 4 below.

Table 3: The 95% Profile-likelihood Confidence Intervals, for n=20

α	r	$[\hat{\alpha}_l, \hat{\alpha}_u]$	CL for α	$[\hat{\beta}_l, \hat{\beta}_u]$	CL for β
0.5	8	[0.32520, 1.32290]	0.99770	[0.58814,2.21117]	1.52356
	4	[0.12235, 0.913881]	0.79153	[0.36406,2.17766]	1.81360
1.0	8	[0.50424, 2.05129]	1.54705	[0.55455,1.75009]	1.19554
	4	[0.18933, 1.41421]	1.22488	[0.41767,1.84438]	1.42671
1.5	8	[0.63331, 2.57639]	1.94308	[0.54927,1.57123]	1.02196
	4	[0.24103, 1.80033]	1.5593	[0.45431,1.70303]	1.24872

From the simulation results in the table 3, it can be observed that for a fixed value of the scale parameter α and overall sample size n, as the effective sample size r increases, the length of the approximate Profile-likelihood confidence interval for the shape parameter β decreases. On the other hand, it can be observed that for a fixed value of the scale parameter α and overall sample size n, as the effective sample size r increases, the length of the approximate profile-likelihood confidence interval for the scale parameter α also increases.

Table 4: The 95% Profile-likelihood Confidence Intervals, for $\boldsymbol{n = 40}$

α	r	$[\hat{\alpha}_l, \hat{\alpha}_u]$	CL for α	$[\hat{\beta}_l, \hat{\beta}_u]$	CL for β
0.5	8	[0.31436, 1.27880]	0.96444	[0.72555, 1.88895]	1.16340
	4	[0.10992, 0.82101]	0.71109	[0.50643, 1.89330]	1.38687
1.0	8	[0.48605, 1.97731]	1.49126	[0.74387, 1.63058]	0.88671
	4	[0.17309, 1.29293]	1.11984	[0.58740, 1.68967]	1.10227
1.5	8	[0.61742, 2.51172]	1.8943	[0.74968, 1.52322]	0.77354
	4	[0.22375, 1.67125]	1.44750	[0.62380, 1.60089]	0.97709

From the simulation results in the table 4, it can be observed that for a fixed value of the scale parameter α and overall sample size n, as the effective sample size r increases, the length of the approximate Profile-likelihood confidence interval for the shape parameter β decreases. On the other hand, it can be observed that for a fixed value of the scale parameter α and overall sample size n, as the effective sample size r increases, the length of the approximate profile-likelihood confidence interval for the scale parameter α also increases.

Furthermore, it can be observed that the 95% profile-likelihood confidence interval results for the scale parameter α and scale parameter β in table 4 are narrower than the 95% profile-likelihood confidence interval results in table 3.

The approximate Profile-likelihood confidence interval for the scale parameter α and shape parameter β vary with respect to the effective sample size r because they are fully conditioned on the shape of the marginal relative likelihood graph of α and β which have their maximum at the MLE's and the MLE's of both parameters α and β are functions of the effective sample size r.

From the results given in tables 1, 2, 3 and 4 above, it can be observed that the 95% profile approximate confidence lengths of α and β for different fixed values of r and n are narrower than those associated with Wald technique. This is because the Wald technique is an approximation which is good for large samples where the central limit theorem is applicable; consequently, the approximate Wald intervals are symmetrical about the MLE which is not appropriate for asymmetric distribution. On the other hand, the Profile-likelihood method is better because it depends on the scaled log-likelihood which is basically the profile-likelihood and it takes care the asymmetric nature of the profile-likelihood function.

A subset of the interval estimates in the above tables for the shape parameter β obtained using the aforementioned techniques on the basis of simulated data, are plotted on the relative profile-likelihood function graph of β in order to illustrate their plausibility. In each graph the horizontal red line segment represents the 95% approximate confidence interval. These plots are given below.

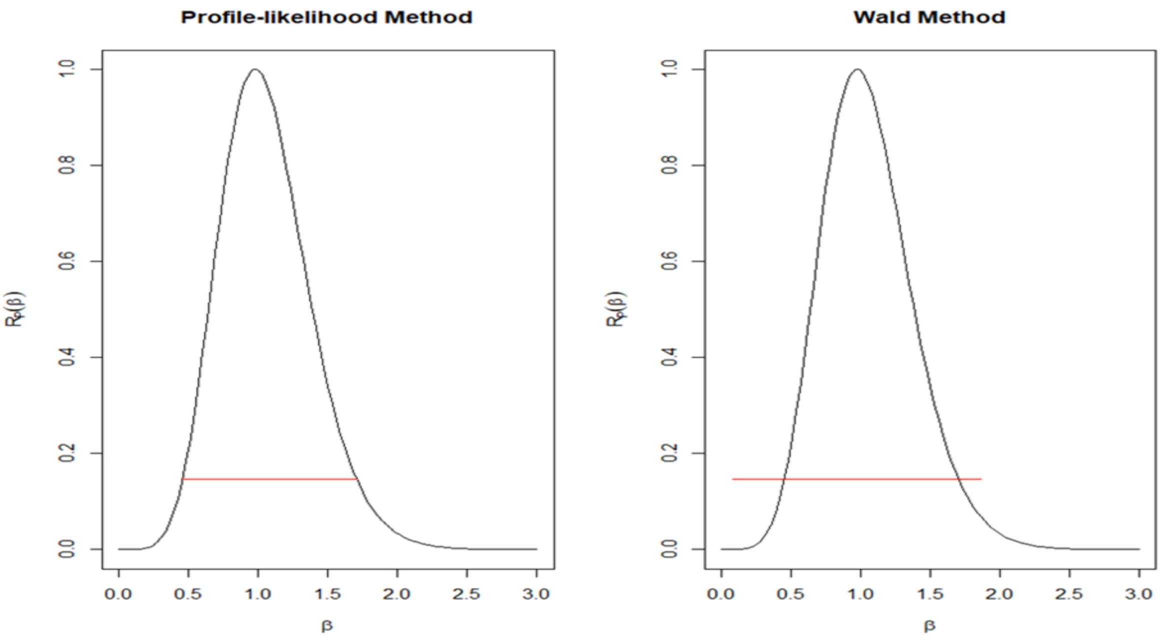

Figure 4.1: Plots of relative profile- likelihood function of β for α=1.5, β=1.5, r=4, n=20

The relative profile- likelihood function of β above, denoted by $R_p(\beta)$, is obtained by maximizing R (α, β) over α fixed as 1.5, r fixed as 4, n fixed as 20, and the graph of $R_p(\beta)$ is plotted for different values of β. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 1.5$, $\beta = 1.5$, $r = 4$, and $n = 20$ are plotted in the graph of $R_p(\beta)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the shape parameter β associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

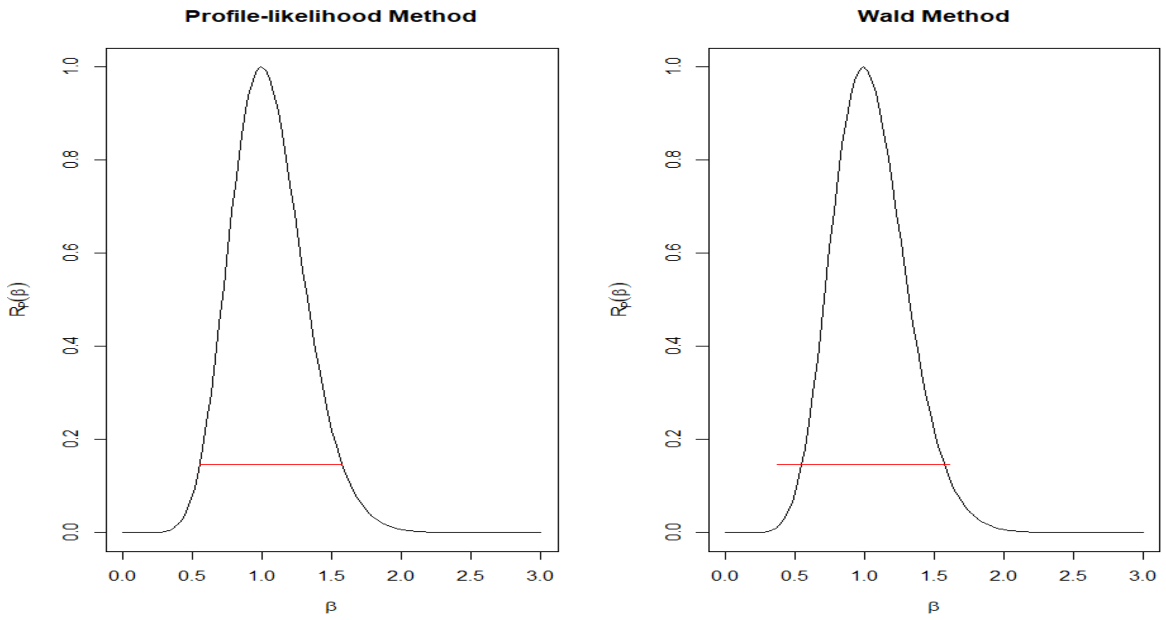

Figure 4.2: Plots of relative profile-likelihood function of β for α=1.5, β=1.5, r=8, n=20

The relative profile- likelihood function of β above, denoted by $R_p(\beta)$, is obtained by maximizing R (α, β) over α fixed as 1.5, r fixed as 8, n fixed as 20, and the graph of $R_p(\beta)$ is plotted for different values of β. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 1.5$, $\beta = 1.5$, $r = 8$, and $n = 20$ are plotted in the graph of $R_p(\beta)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the shape parameter β associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

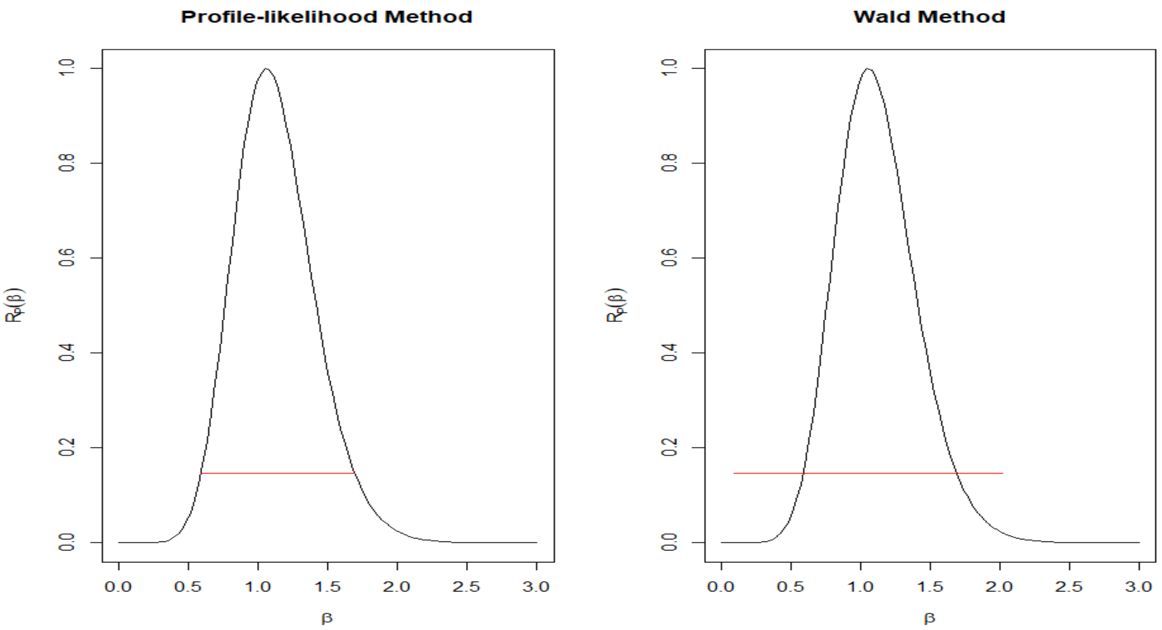

Figure 4.3: Plots of relative profile- likelihood function of β for α=1, β=1.5, r=4, n=40

The relative profile- likelihood function of β above, denoted by $R_p(\beta)$, is obtained by maximizing R (α, β) over α fixed as 1, r fixed as 4, n fixed as 40, and the graph of $R_p(\beta)$ is plotted for different values of β. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 1$, $\beta = 1.5$, $r = 4$, and $n = 40$ are plotted in the graph of $R_p(\beta)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the shape parameter β associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

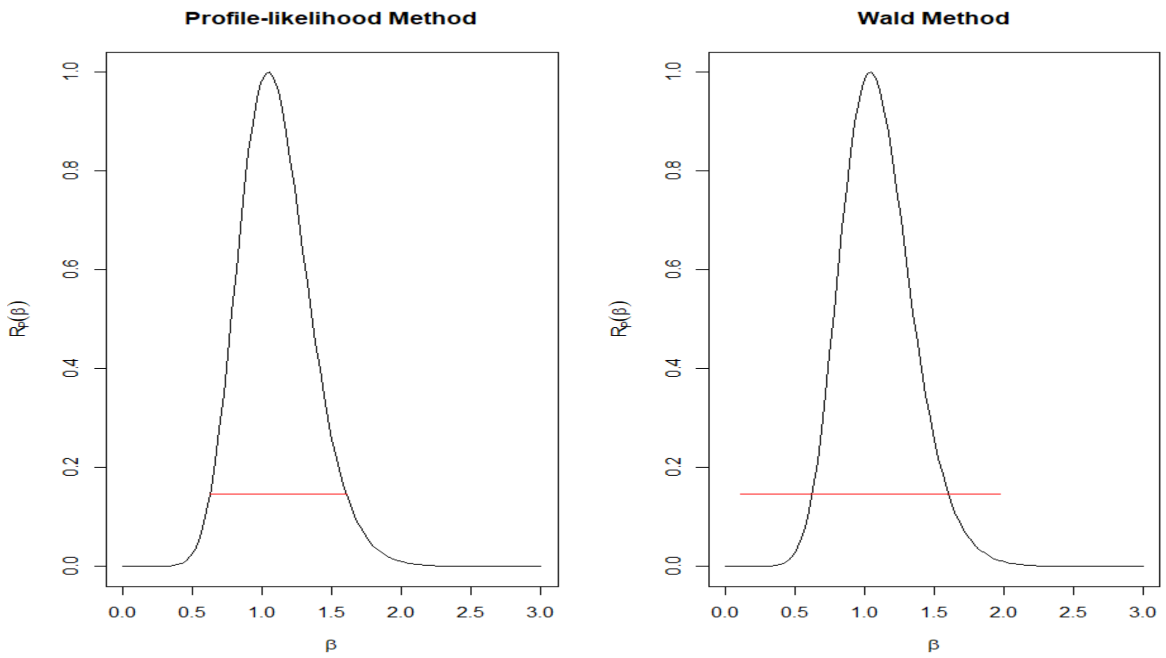

Figure 4.4: Plots of relative profile-likelihood function of β for α=1.5, β=1.5, r=4, n=40

The relative profile- likelihood function of β above, denoted by $R_p(\beta)$, is obtained by maximizing R (α, β) over α fixed as 1.5, r fixed as 4, n fixed as 40, and the graph of $R_p(\beta)$ is plotted for different values of β. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 1.5$, $\beta = 1.5$, $r = 4$, and $n = 40$ are plotted in the graph of $R_p(\beta)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the shape parameter β associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

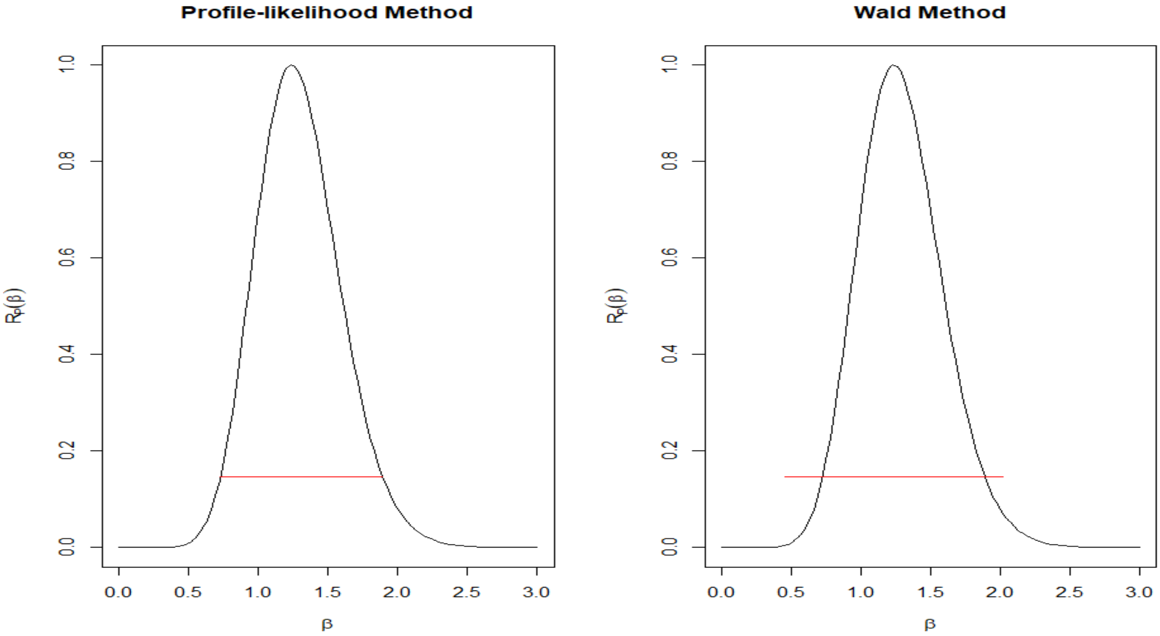

Figure 4.5: Plots of relative profile- likelihood function of β for α=0.5, β=1.5, r=8, n=40

The relative profile- likelihood function of β above, denoted by $R_p(\beta)$, is obtained by maximizing R (α, β) over α fixed as 0.5, r fixed as 8, n fixed as 40, and the graph of $R_p(\beta)$ is plotted for different values of β. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 0.5$, $\beta = 1.5$, $r = 8$, and $n = 40$ are plotted in the graph of $R_p(\beta)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the shape parameter β associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

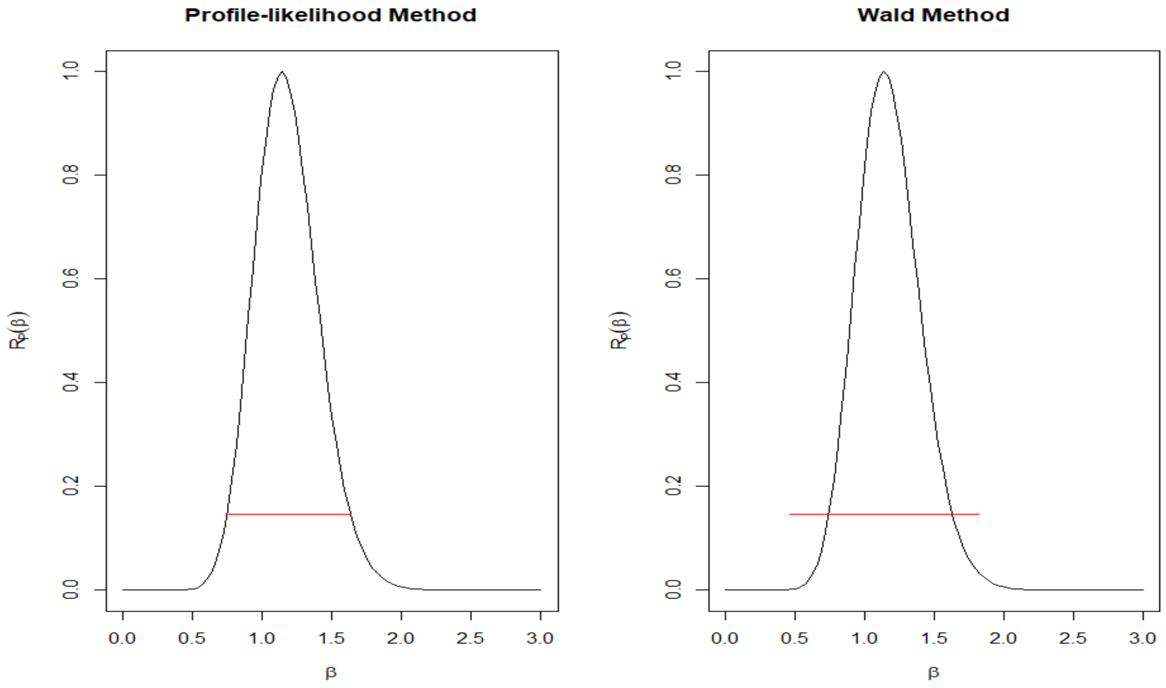

Figure 4.6: Plots of relative profile-likelihood function of β for α=1, β=1.5, r=8, n=40

The relative profile- likelihood function of β above, denoted by $R_p(\beta)$, is obtained by maximizing R (α, β) over α fixed as 1, r fixed as 8, n fixed as 40, and the graph of $R_p(\beta)$ is plotted for different values of β. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 1$, $\beta = 1.5$, $r = 8$, and $n = 40$ are plotted in the graph of $R_p(\beta)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the shape parameter β associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

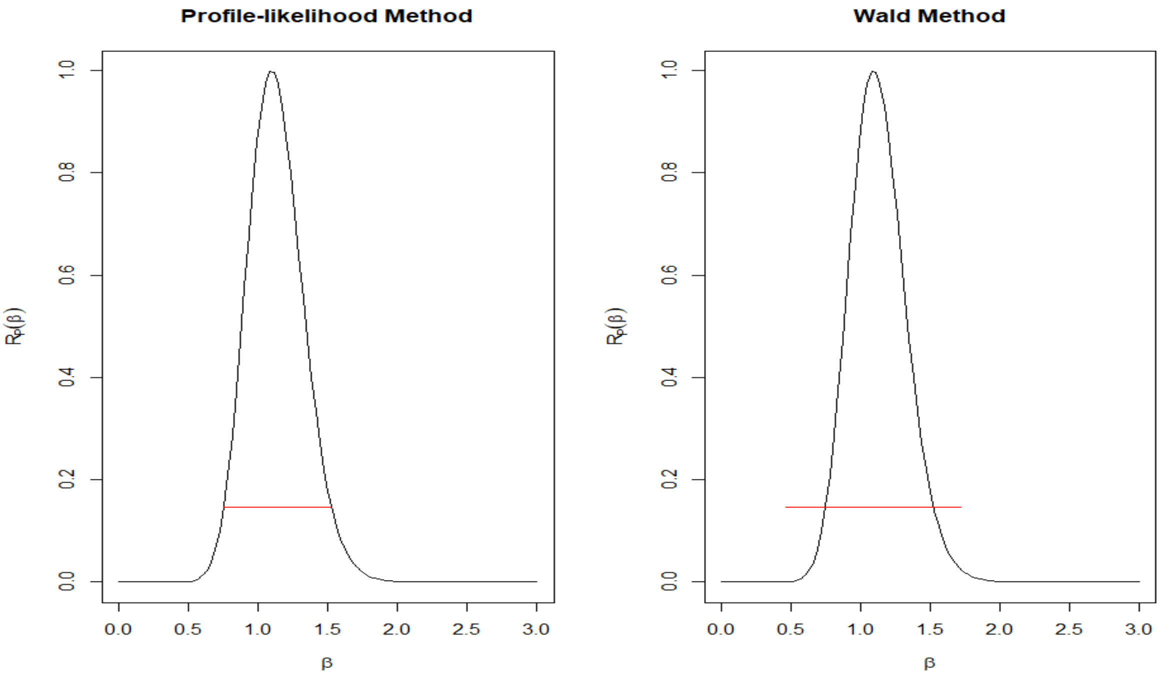

Figure 4.7: Plots of relative profile-likelihood function of β for α=1 .5, β=1.5, r=8, n=40

The relative profile- likelihood function of β above, denoted by $R_p(\beta)$, is obtained by maximizing R (α, β) over α fixed as 1.5, r fixed as 8, n fixed as 40, and the graph of $R_p(\beta)$ is plotted for different values of β. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 1.5$, $\beta = 1.5$, $r = 8$, and $n = 40$ are plotted in the graph of $R_p(\beta)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the shape parameter β associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

It is evident from the figures 4.1 to 4.7 above the interval estimates of the shape parameter β associated with the profile-likelihood method contain plausible values as compared to those associated with the Wald technique irrespective of the sample size n.

Similarly, a subset of interval estimates for the scale parameter α obtained using the aforementioned techniques, are plotted on the relative profile-likelihood function graph of α in order to illustrate their plausibility. The horizontal red line segment represents 95% approximate confidence interval. These plots are given below.

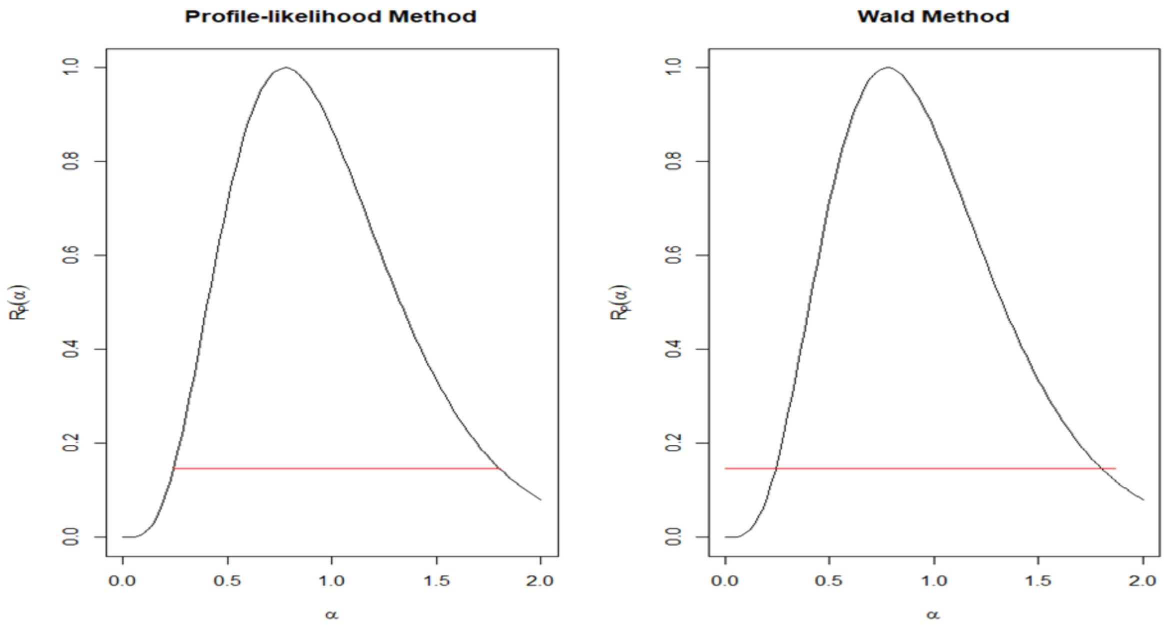

Figure 4.8: Plots of relative profile-likelihood function of $\boldsymbol{\alpha}$ for α=1 .5, β=1.5, r=4, n=20

The relative profile- likelihood function of α above, denoted by $R_p(\alpha)$, is obtained by maximizing R (α, β) over β fixed as 1.5, r fixed as 4, n fixed as 20, and the graph of $R_p(\alpha)$ is plotted for different values of α. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 1.5$, $\beta = 1.5$, $r = 4$, and $n = 20$ are plotted in the graph of $R_p(\alpha)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the scale parameter α associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

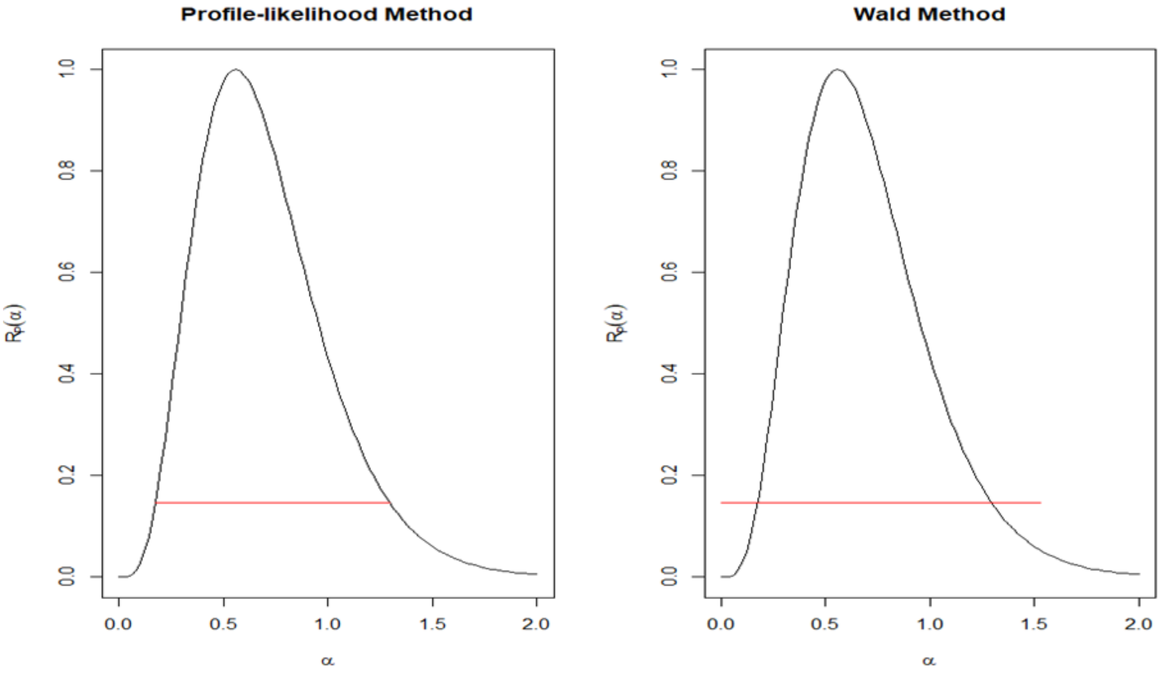

Figure 4.9: Plots of relative profile- likelihood function of α for α=1, β=1.5, r=4, n=40

The relative profile- likelihood function of α above, denoted by $R_p(\alpha)$, is obtained by maximizing R (α, β) over β fixed as 1.5, r fixed as 4, n fixed as 40, and the graph of $R_p(\alpha)$ is plotted for different values of α. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 1$, $\beta = 1.5$, $r = 4$, and $n = 40$ are plotted in the graph of $R_p(\alpha)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the scale parameter α associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

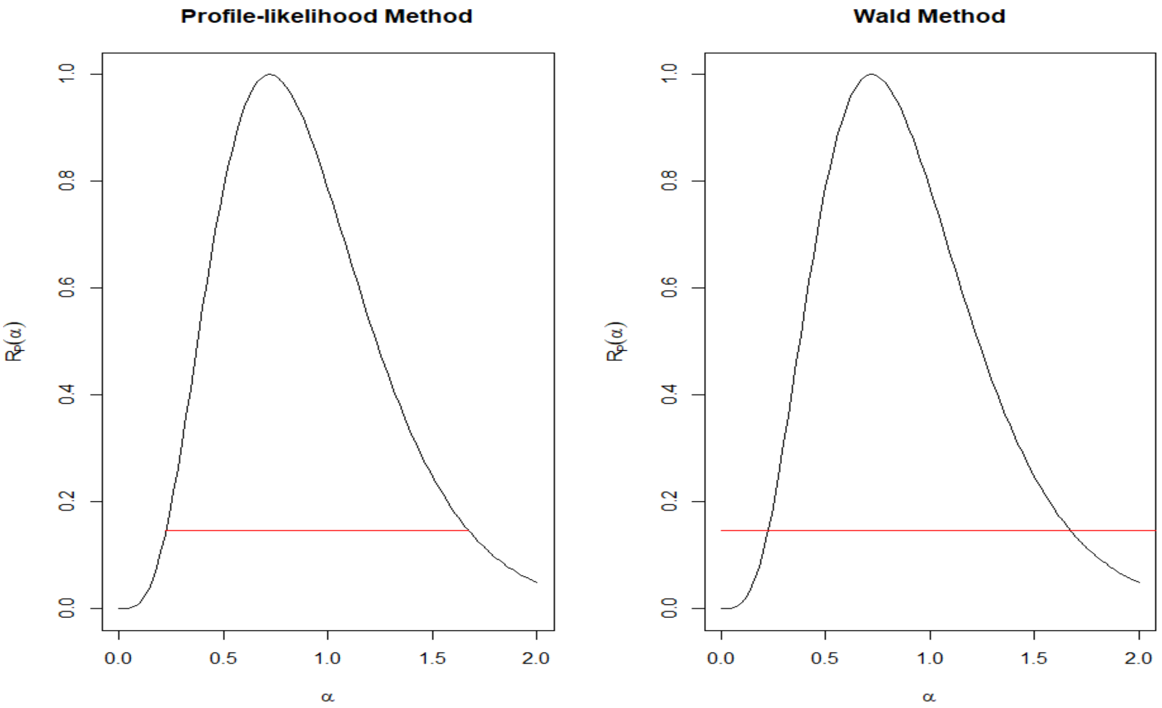

Figure 4.10: Plots of relative profile- likelihood function of α for α=1 .5, β=1.5, r=4, n=40

The relative profile- likelihood function of α above, denoted by $R_p(\alpha)$, is obtained by maximizing R (α, β) over β fixed as 1.5, r fixed as 4, n fixed as 40, and the graph of $R_p(\alpha)$ is plotted for different values of α. The interval estimate associated with profile-likelihood and Wald method when $\alpha = 1.5$, $\beta = 1.5$, $r = 4$, and $n = 40$ are plotted in the graph of $R_p(\alpha)$ in order to illustrate their plausibility. The horizontal red line length shows the length of the confidence interval estimate associated with both methods. From the above two graphs, it can be seen that, the interval estimate of the scale parameter α associated with the profile-likelihood method is narrower as compared to the one associated with the Wald technique.

It is evident from the figures 4.8 to 4.10 above, the length of the interval estimates of the scale parameter α associated with the profile-likelihood method are narrower as compared to those associated with the Wald technique irrespective of the sample size n.

4.4 The Contour Regions for (α, β) for different values of r and n

The likelihood function of α and β does not factor orthogonally and hence contour plots are used to check the overall accuracy of the point estimates of α and β computed using the method of maximum likelihood. Narrow contour plots are associated with better accuracy on the maximum likelihood estimates while wide contour plots are associated with poor accuracy on the estimates of parameters. The black star denotes the location of the point estimates while the red star denotes the location of the true values. The 3.6%, 14.7% and 25.8% likelihood contour regions were plotted for different set of values of α and β using the joint relative likelihood function of α and β; they are, respectively, approximate 96.4% ,83.3%, and 74.2% confidence regions(Kabfleisch, 1985).These plots are given below.

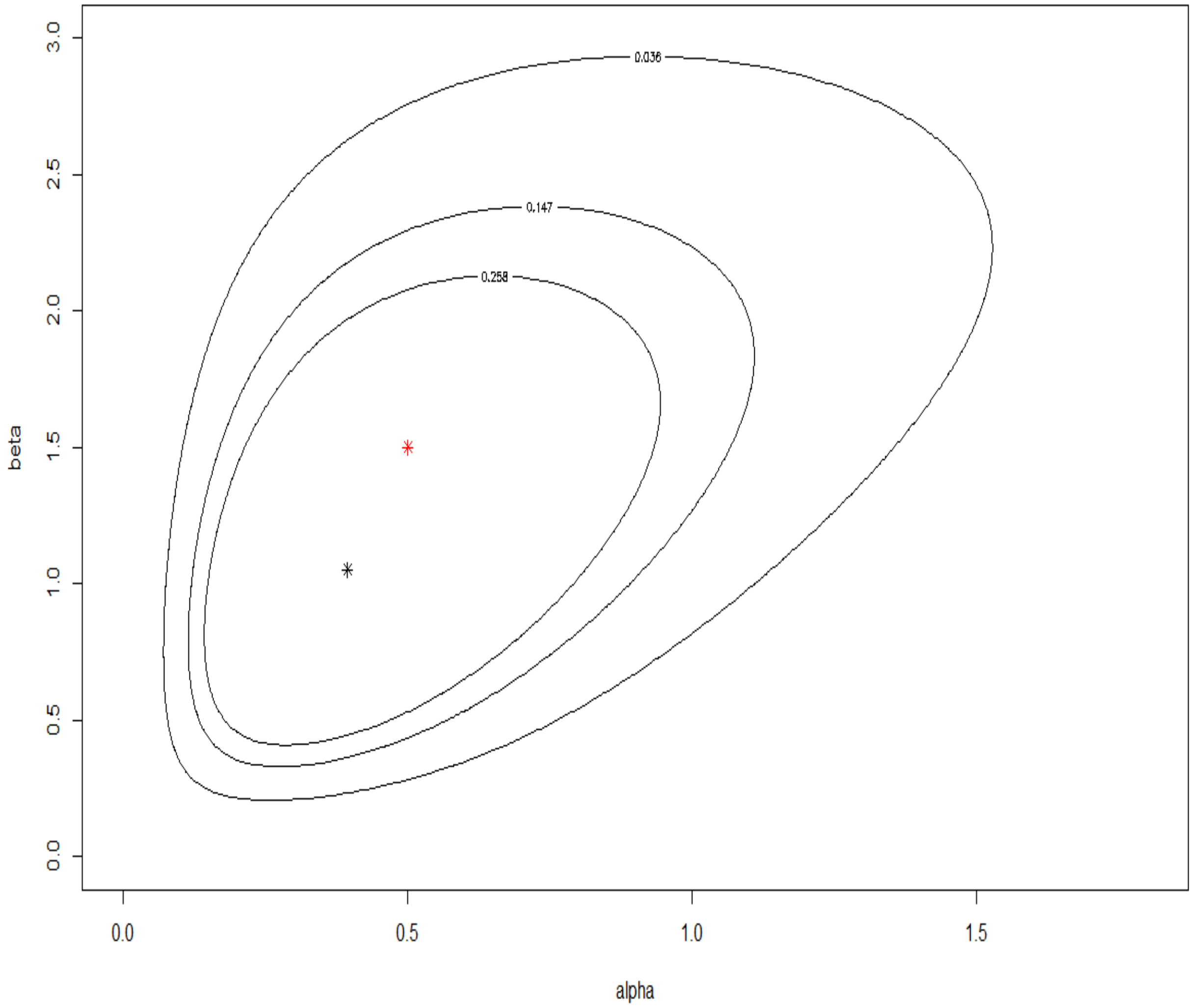

Figure 4.11: Contour likelihood regions for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ when $\boldsymbol{\alpha}$=0.5, $\boldsymbol{\beta}$=1.5, $\boldsymbol{r}$=4, $\boldsymbol{n}$=20

Profile- likelihood technique seems accurate in jointly estimating the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the 2-parameter Weibull distribution because the true parameters are contained in all the contour likelihood regions when $\boldsymbol{\alpha}$ is fixed as 0.5, $\boldsymbol{\beta}$ as 1.5, effective sample size $\boldsymbol{r}$ as 4, and the overall sample size $\boldsymbol{n}$ as 20. The parameter estimates are relatively close to the true values.

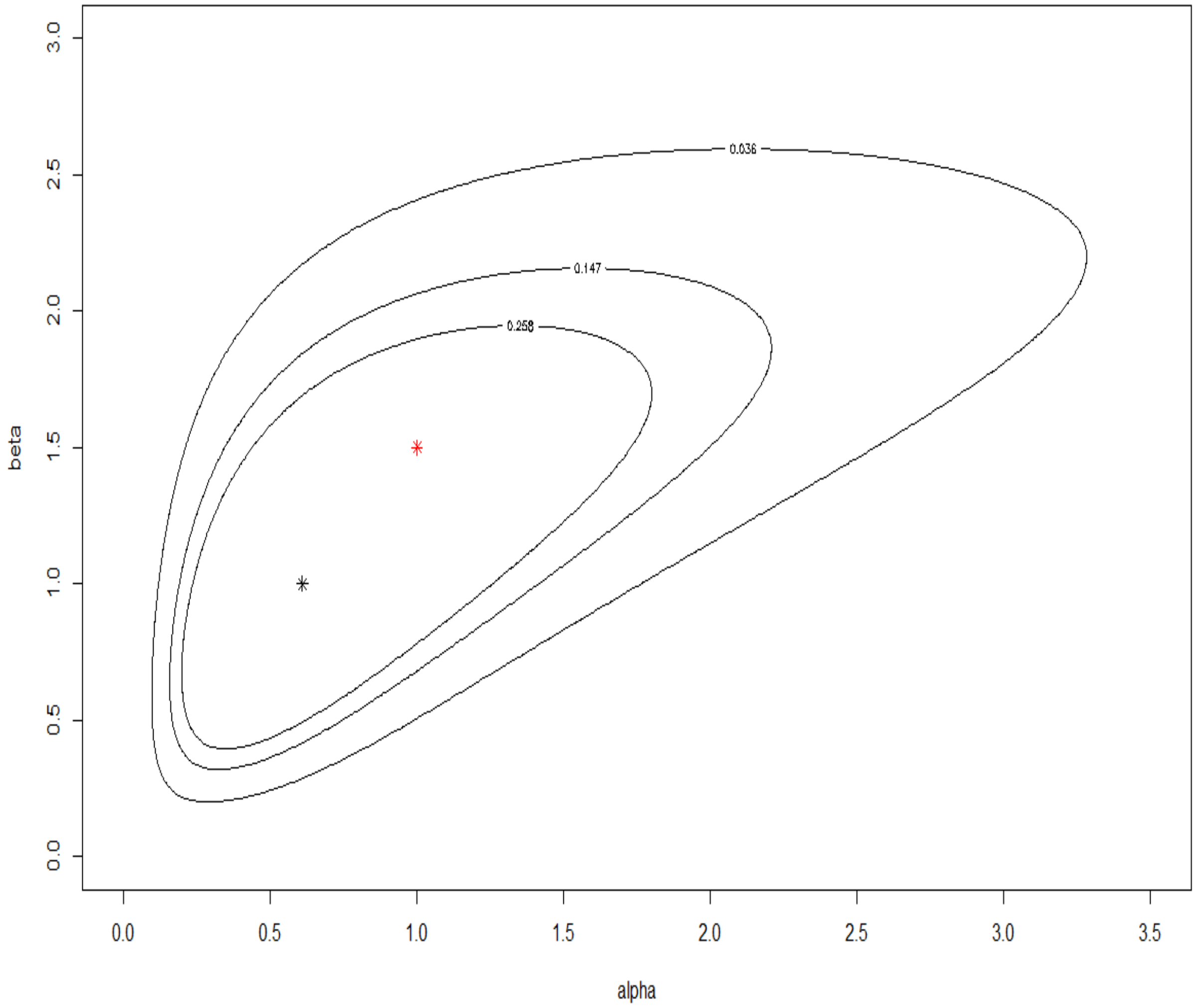

Figure 4.12: Contour likelihood regions for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ when $\boldsymbol{\alpha}$=1, $\boldsymbol{\beta}$=1.5, $\boldsymbol{r}$=4, $\boldsymbol{n}$=20

Profile- likelihood technique seems accurate in jointly estimating the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the 2-parameter Weibull distribution because the true parameters are contained in all the contour likelihood regions when $\boldsymbol{\alpha}$ is fixed as 1, $\boldsymbol{\beta}$ as 1.5, effective sample size $\boldsymbol{r}$ as 4, and the overall sample size $\boldsymbol{n}$ as 20. The parameter estimates are still relatively close to the true values.

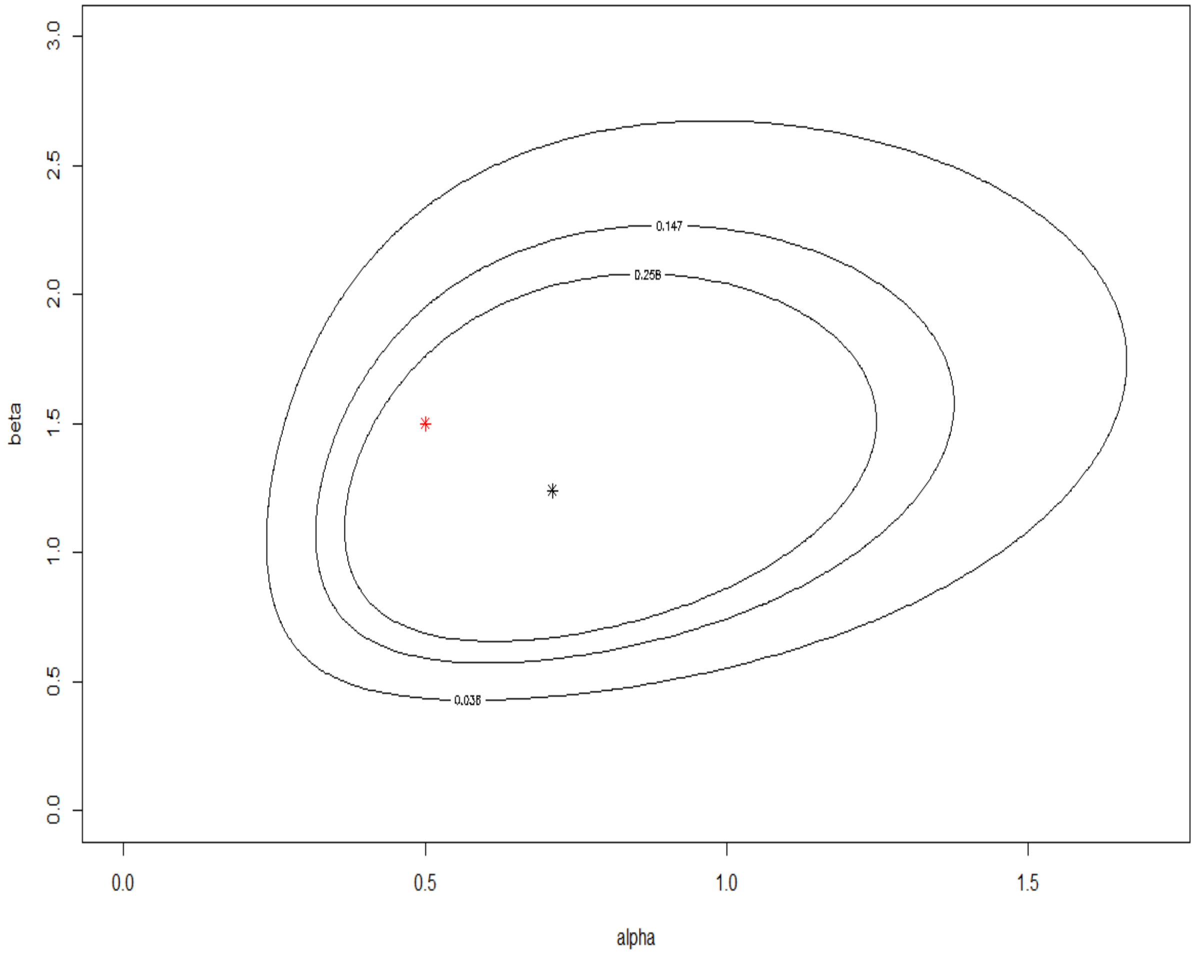

Figure 4.13: Contour likelihood regions for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ when $\boldsymbol{\alpha}$=0.5, $\boldsymbol{\beta}$=1.5, $\boldsymbol{r}$=8, $\boldsymbol{n}$=20

Profile- likelihood technique seems accurate in jointly estimating the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the 2-parameter Weibull distribution because the true parameters are contained in all the contour likelihood regions when $\boldsymbol{\alpha}$ is fixed as 0.5, $\boldsymbol{\beta}$ as 1.5, effective sample size $\boldsymbol{r}$ as 8, and the overall sample size $\boldsymbol{n}$ as 20. The parameter estimates are still relatively close to the true values.

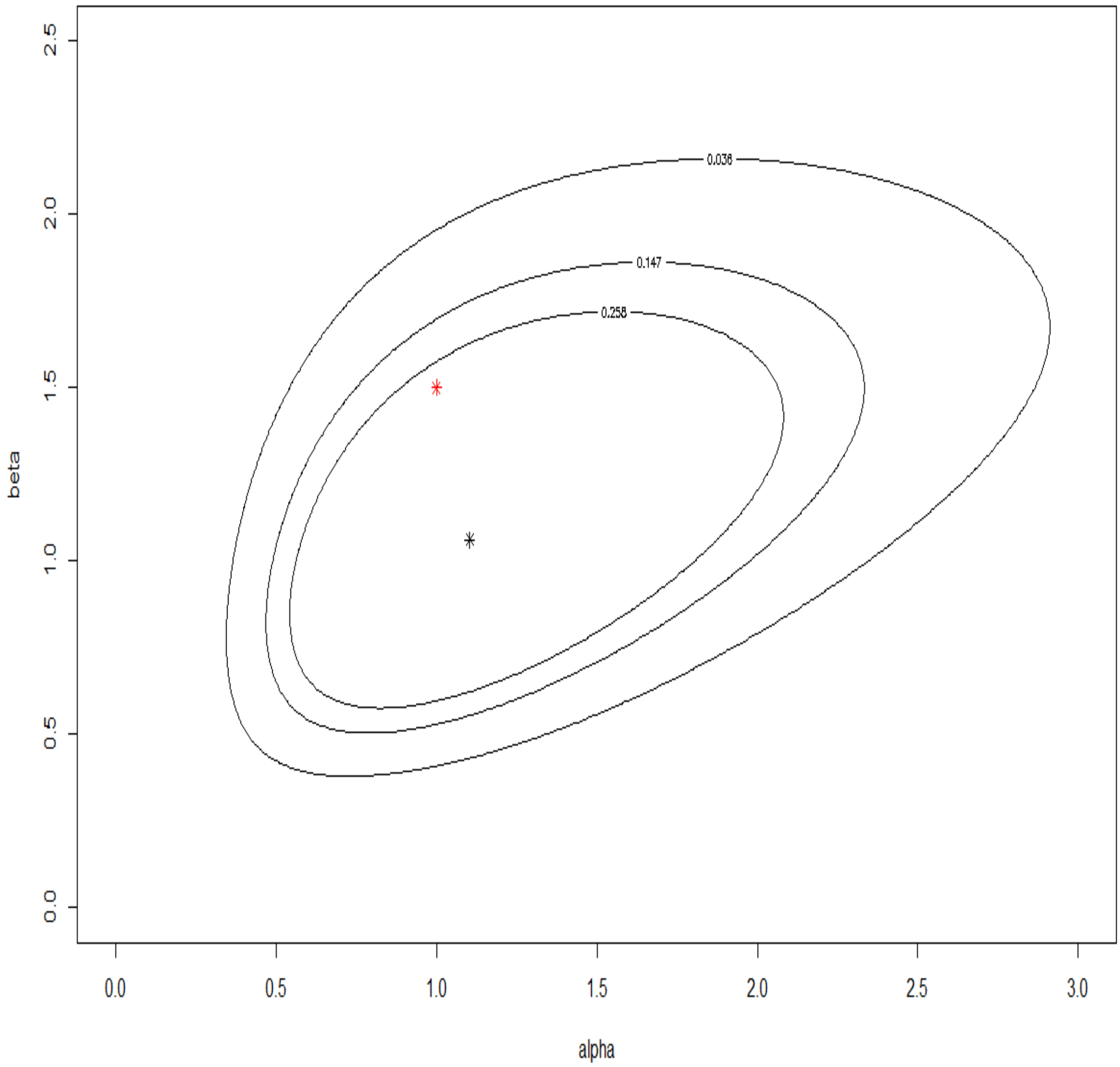

Figure 4.14: Contour likelihood regions for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ when $\boldsymbol{\alpha}$=1, $\boldsymbol{\beta}$=1.5, $\boldsymbol{r}$=8, $\boldsymbol{n}$=20

Profile-likelihood technique still seems accurate in jointly estimating the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the 2-parameter Weibull distribution because the true parameters are contained in all the contour likelihood regions when $\boldsymbol{\alpha}$ is fixed as 1, $\boldsymbol{\beta}$ as 1.5, effective sample size $\boldsymbol{r}$ as 8, and the overall sample size $\boldsymbol{n}$ as 20. The parameter estimates are relatively close to the true values.

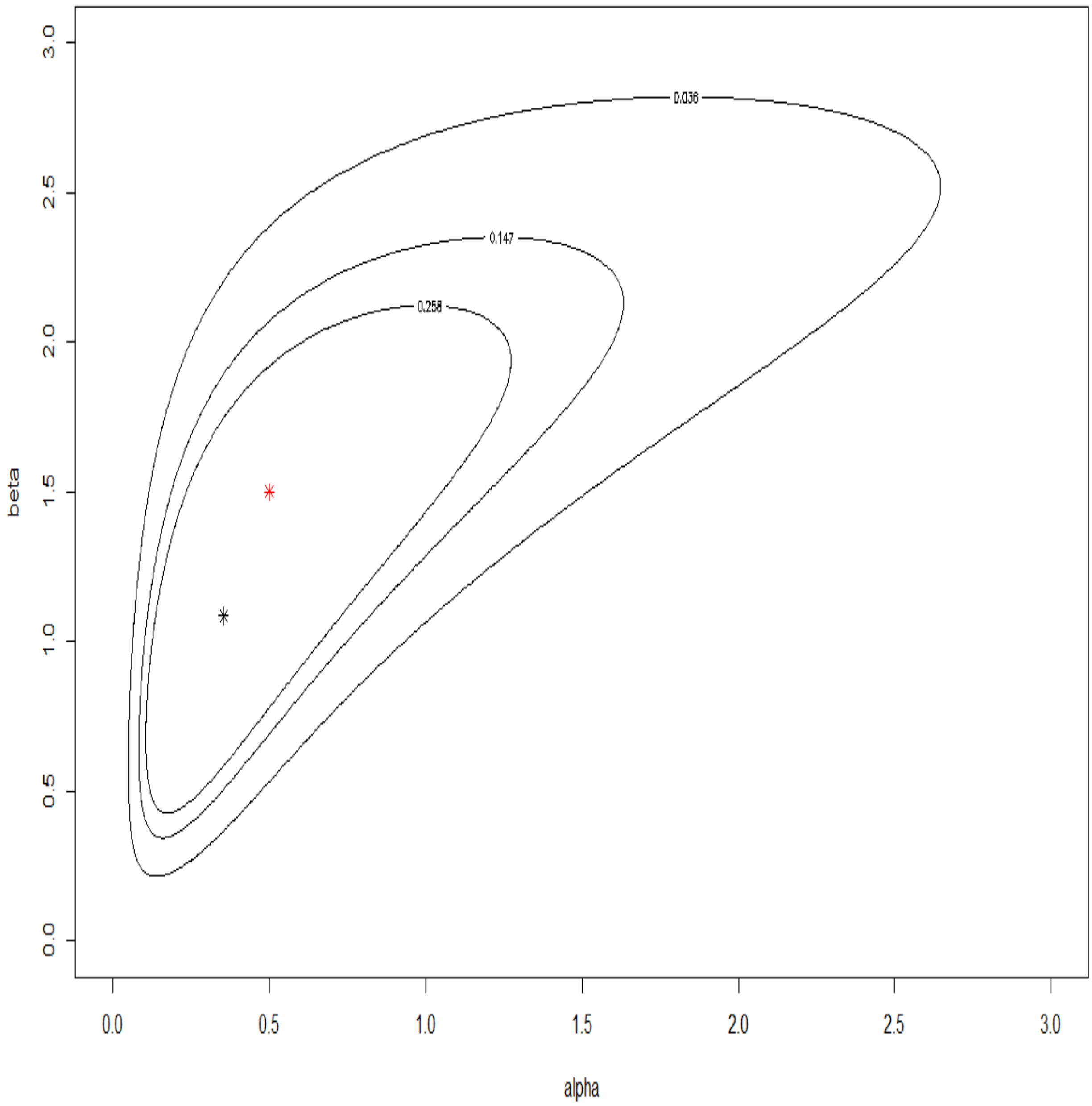

Figure 4.15: Contour likelihood regions for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ when $\boldsymbol{\alpha}$=0.5, $\boldsymbol{\beta}$=1.5, $\boldsymbol{r}$=4, $\boldsymbol{n}$=40

Profile-likelihood technique seems accurate in jointly estimating the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the 2-parameter Weibull distribution because the true parameters are contained in all the contour likelihood regions when $\boldsymbol{\alpha}$ is fixed as 0.5, $\boldsymbol{\beta}$ as 1.5, effective sample size $\boldsymbol{r}$ as 4, and the overall sample size $\boldsymbol{n}$ as 40. The parameter estimates are still relatively close to the true value.

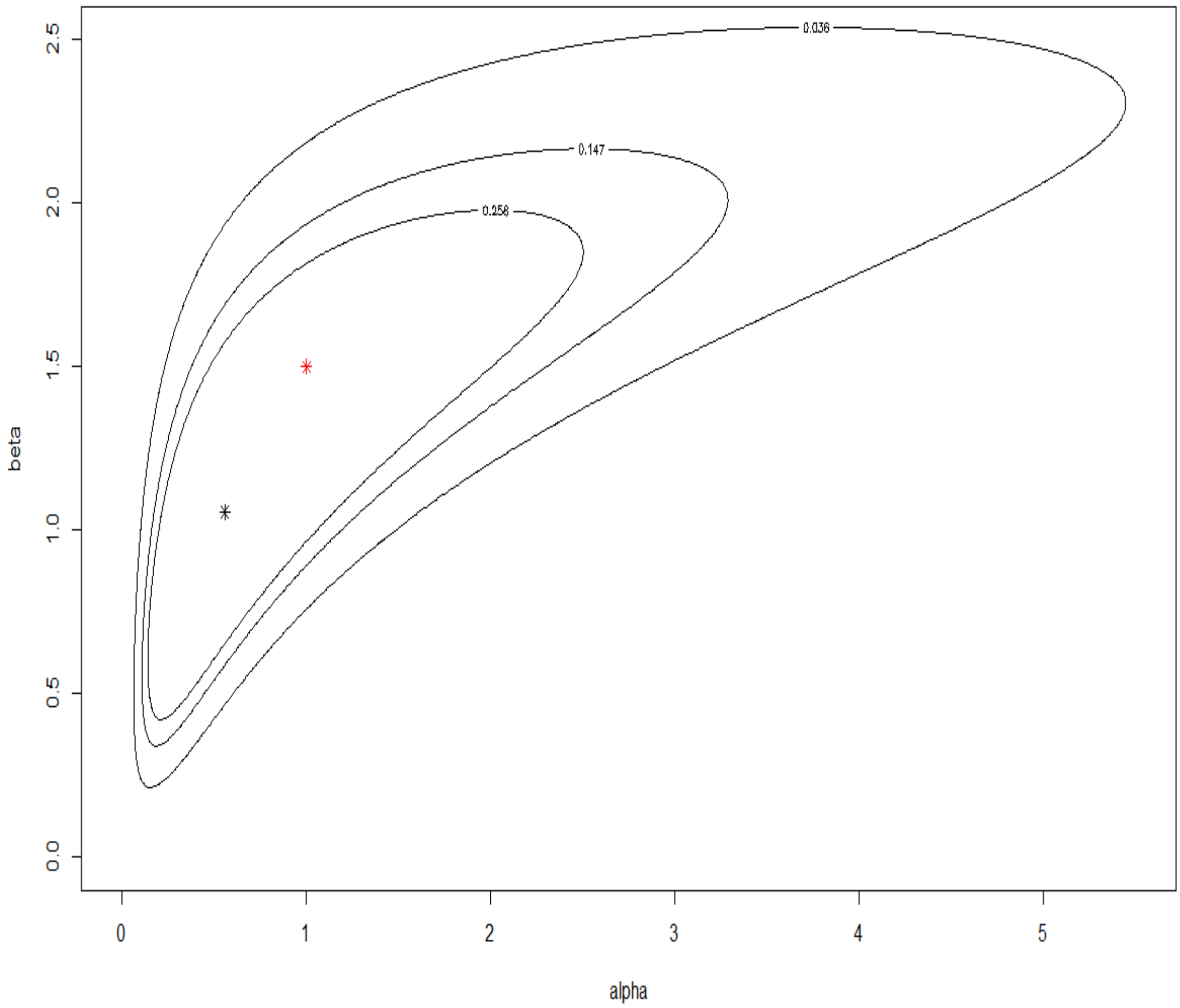

Figure 4.16: Contour likelihood regions for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ when $\boldsymbol{\alpha}$=1, $\boldsymbol{\beta}$=1.5, $\boldsymbol{r}$=4, $\boldsymbol{n}$=40

Profile-likelihood technique still seems accurate in jointly estimating the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the 2-parameter Weibull distribution because the true parameters are contained in all the contour likelihood regions when $\boldsymbol{\alpha}$ is fixed as 1, $\boldsymbol{\beta}$ as 1.5, effective sample size $\boldsymbol{r}$ as 4, and the overall sample size $\boldsymbol{n}$ as 40. The parameter estimates are relatively close to the true values.

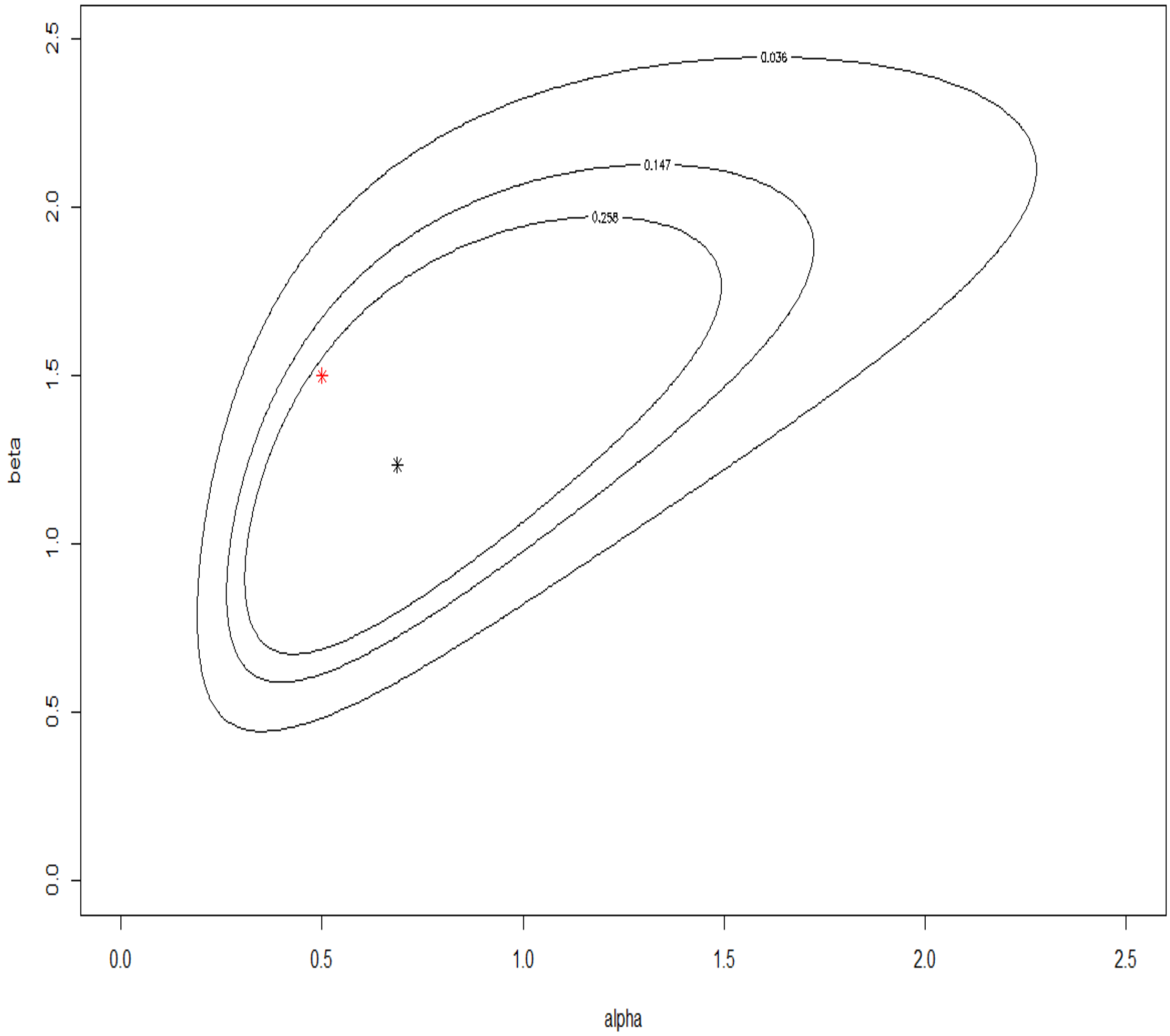

Figure 4.17: Contour likelihood regions for α and β when α=0.5, β=1.5, r=8, n=40

Profile-likelihood technique seems accurate in jointly estimating the unknown parameters α and β of the 2-parameter Weibull distribution because the true parameters are contained in all the contour likelihood regions when α is fixed as 0.5, β as 1.5, effective sample size r as 8, and the overall sample size n as 40. The parameter estimates are still relatively close to the true values.

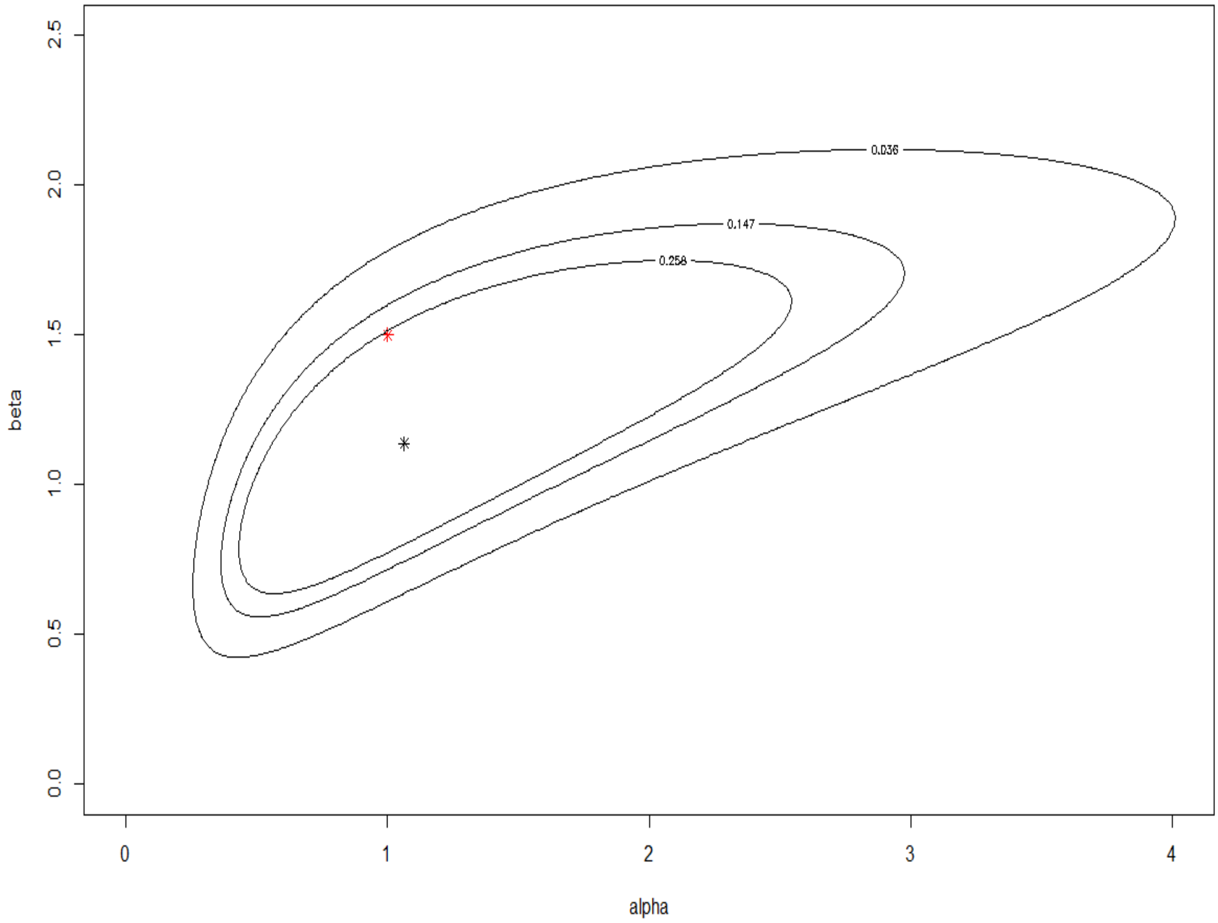

Figure 4.18: Contour likelihood regions for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ when $\boldsymbol{\alpha}$=1, $\boldsymbol{\beta}$=1.5, $\boldsymbol{r}$=8, $\boldsymbol{n}$=40

Profile-likelihood technique still seems accurate in jointly estimating the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the 2-parameter Weibull distribution because the true parameters are contained in all the contour likelihood regions when $\boldsymbol{\alpha}$ is fixed as 0.5, $\boldsymbol{\beta}$ as 1.5, effective sample size $\boldsymbol{r}$ as 8, and the overall sample size $\boldsymbol{n}$ as 40. The parameter estimates are relatively close to the true values.

The above likelihood contour plots provide a description of the graph of the joint likelihood function and indicate the regions of plausibility of α and β jointly. Values of α and β outside 3.6% likelihood contour region are relatively implausible since their relative likelihoods are less than 3.6% of the maximum. Profile- likelihood technique seems accurate in jointly estimating the unknown parameters α and β of the 2-parameter Weibull distribution because the true parameters are contained in all the contour likelihood regions, even within the approximate 74.2% confidence region. The parameter estimates are relatively close to the true values irrespective of the effective sample size r and overall sample size n

4.4 The Coverage Probability Results

The 95% coverage probability results for the interval estimates of the 2- parameter Weibull distribution associated with Wald and profile-likelihood technique under type-2 censoring scheme are given in tables 5, 6, 7 and 8.

Table 5: The 95% Wald Approximate Coverage Probabilities for $\mathrm{n} = 20$

$\boldsymbol{\alpha}$	r	$\boldsymbol{CP_{\alpha}}$	$\boldsymbol{CP_{\beta}}$
0.5	8	0.845	0.956
	4	0.910	0.976
1.0	8	0.798	0.912
	4	0.895	0.968
1.5	8	0.682	0.858
	4	0.848	0.962

From the results in table 5, it can be observed that most of the coverage probabilities for the shape parameter β are greater than or relatively close to the nominal coverage probability (0.95). Similarly, it can also be observed that, that the coverage probabilities of the scale parameter α are less than the nominal coverage probability (0.95).

Table 6: The 95% Wald Approximate Coverage Probabilities for $\boldsymbol{n = 40}$

$\boldsymbol{\alpha}$	**r**	$\boldsymbol{CP_\alpha}$	$\boldsymbol{CP_\beta}$
0.5	8	0.915	0.958
	4	0.948	0.977
1.0	8	0.844	0.942
	4	0.920	0.976
1.5	8	0.767	0.918
	4	0.897	0.976

From the results in table 6, it can be observed that most of the coverage probabilities for the shape parameter β are greater than or relatively close to the nominal coverage probability (0.95). On the other hand, it can be observed that the coverage probabilities for the scale parameter α are less than or relatively close to the nominal coverage probability (0.95). Based on the coverage probability results given in tables 5 and 6, it can be observed that Wald coverage probabilities improve when sample size n is increased.

Table 7: The 95 % Profile-Likelihood Coverage Probabilities for $\boldsymbol{n = 20}$

α	r	CP_α	CP_β
0.5	8	0.873	0.923
	4	0.949	0.947
1.0	8	0.927	0.945
	4	0.958	0.966
1.5	8	0.682	0.881
	4	0.860	0.937

From results given in table 7, it can be observed that most of the coverage probabilities for the shape parameter β are very close to the nominal coverage probability (0.95) and the coverage probabilities for the scale parameter α are less than or relatively close to the nominal coverage probability.

Table 8: The 95 % Profile-Likelihood Coverage Probabilities for $n = 40$

α	r	CP_α	CP_β
0.5	8	0.955	0.961
	4	0.945	0.924
1.0	8	0.886	0.889
	4	0.892	0.937
1.5	8	0.674	0.817
	4	0.755	0.905

From the results given in table 8, it can be observed the coverage probabilities for the shape parameter β are either less than, greater than or relatively close to the nominal coverage probability (0.95). Similarly, it can be observed that most of the coverage probabilities for the scale parameter α are less than the nominal coverage probability. This is because the MLE for α is a function of β as evidenced in equation (3.6) in section 3.2.

Moreover, based on the coverage probability simulated results in tables 7 and 8, it can be observed that most of the profile-likelihood coverage probabilities for both parameters are close to the nominal coverage probability (0.95) when the sample size n is small. That is, when $n = 20$.

4.5 Real Data Results

Using the real type-2 censored data provided in section 3.4, the MLE's of the parameters α and β of the 2-parameter Weibull distribution under type-2 censoring scheme are obtained as 0.3392821 and 4.4127589 respectively. Using, the Wald method, the 95 % approximate confidence intervals for the scale parameter α and shape parameter β are obtained as [0.0485729, 0.629991] and [2.195963, 6.629554] respectively. Further, using the profile-likelihood technique, the 95% approximate confidence intervals for the scale parameter α and shape parameter β are obtained as [0.1631579, 0.6113762] and [2.724605, 6.025074]. Based on these results obtained using real data, it can be observed that the interval estimates obtained by the profile-likelihood method are narrower as compared to those obtained using the Wald technique.

Using the real data, interval estimates obtained using both approaches, are plotted on the relative profile-likelihood function graph of α and β in order to illustrate their plausibility. The horizontal red line denotes 14.7% percentile which is equivalent to 95% confidence interval. These plots are given below.

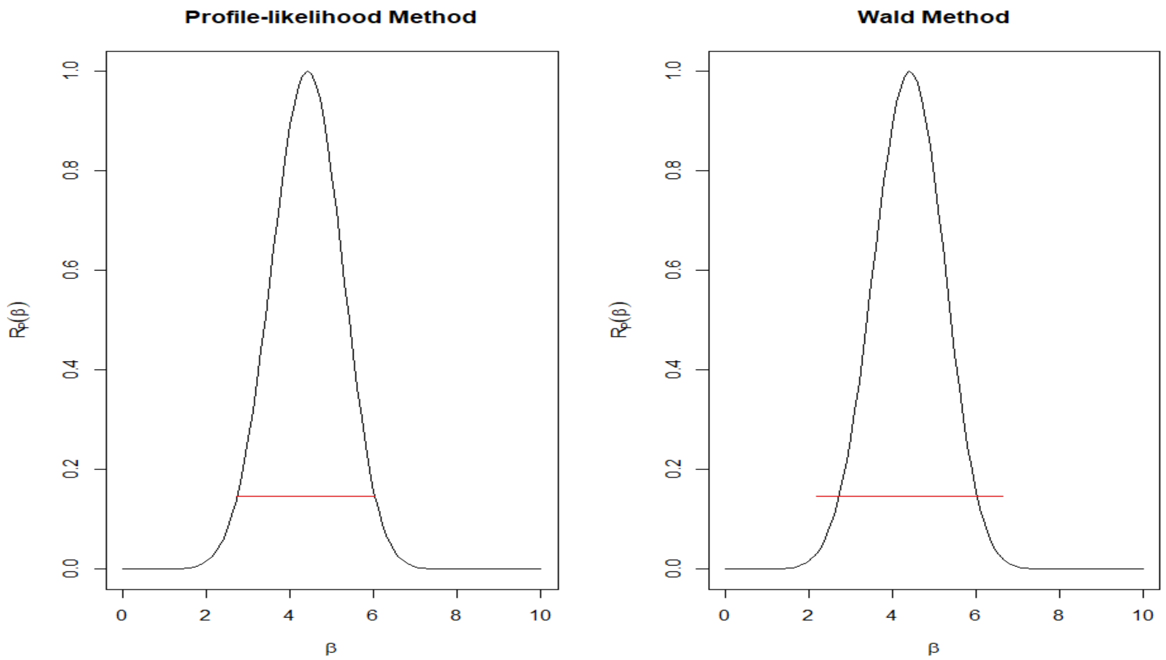

Figure 4.19: Plots of relative profile- likelihood function of β for r=9, n=12

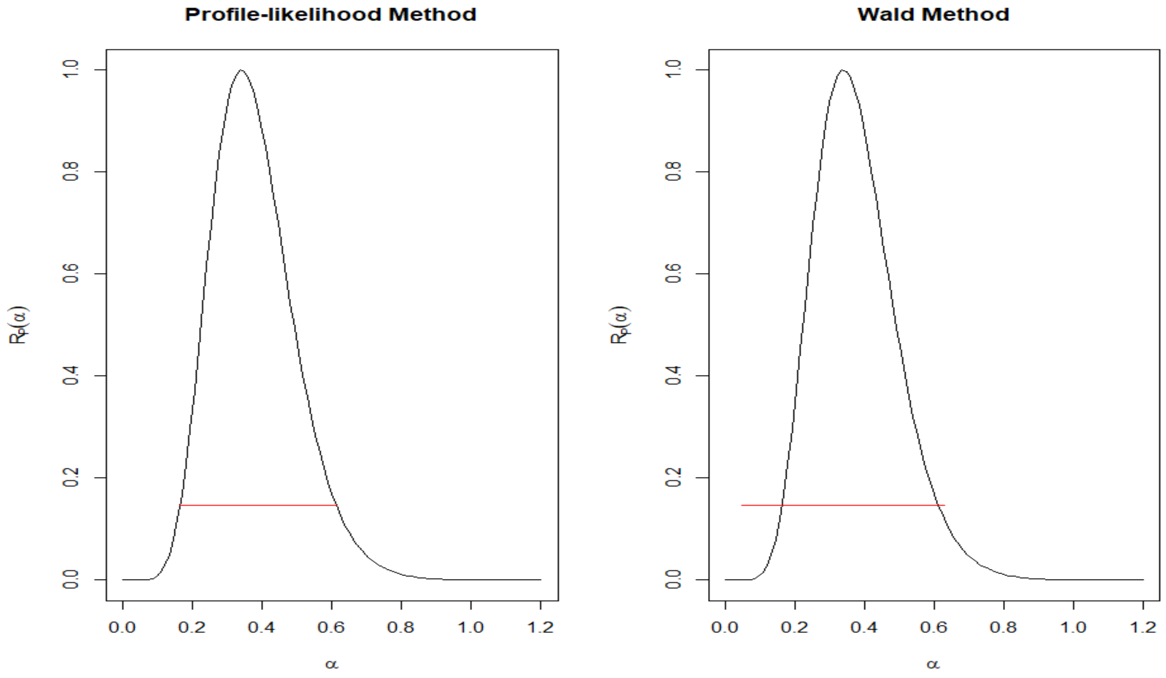

Figure 4.20: Plots of relative profile-likelihood function of α for r=9, n=12

It is evident from the plots above, the interval estimates obtained by the profile-likelihood method are narrower as compared to those obtained using the Wald technique.

Using real data, joint estimation for the aforementioned parameters was also done using the joint relative likelihood function in which, the 90%, 95%, and 99% contour likelihood regions for α and β were plotted. The plot is given below**.**

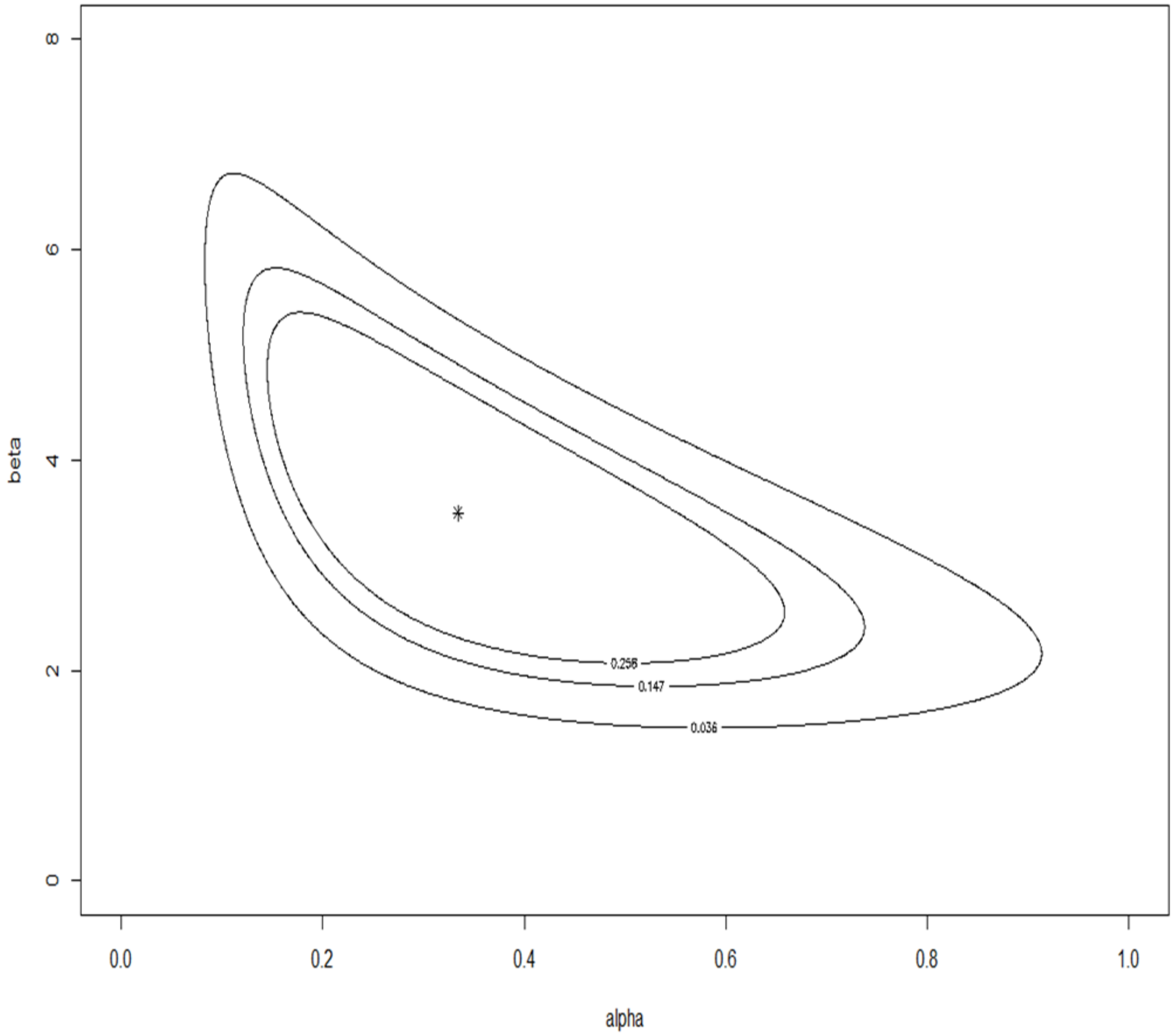

Figure 4.21: Contour likelihood regions for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ when $\boldsymbol{r}$=9, $\boldsymbol{n}$=12

From the above likelihood contour plot, it can be observed that the profile-likelihood technique provides better accuracy in jointly estimating the unknown parameters α and β of the 2-parameter Weibull distribution because even within the 90% contour likelihood region, the point estimate for α and β denoted by black star is still contained inside.

CHAPTER FIVE

CONCLUSIONS AND RECOMMENDATIONS

5.1 Summary

Previous research works have done interval estimation of a 2-parameter Weibull distribution based on type – 2 censored data when the sample size is large by employing Wald method. This method has yielded plausible results when the sample size is large and implausible results for small sample size. This research therefore considered the method of profile-likelihood as an efficient technique to use when constructing approximate confidence intervals for parameters the 2-paramater Weibull distribution based on small type-2 censored samples.

Simulated as well as real life data were used in this study. Interval estimates obtained suggest that the profile-likelihood method is superior over Wald technique if the effective sample size is small. The narrower confidence intervals associated with profile-likelihood method justify the estimates are good and adorable.

5.2 Conclusions

The following conclusions were derived from study:

i. Profile-likelihood method is more efficient to Wald method in constructing approximate confidence interval for the scale parameter α and the shape parameter β for the two parameter Weibull distribution based on both simulated and real type-2 censored data.

ii. The confidence lengths of interval estimates for the scale parameter α and shape parameter β obtained by profile-likelihood technique are relatively narrower than those associated with the Wald technique for both small and large sample size. Notably, for small sample size $n = 20$, most of the coverage probability results for the parameters α and β associated with the Wald method are not very stable because they are below the nominal coverage probability (0.95). On the other hand, for the large sample size $= 40$, most of the coverage probability results for the parameters α and β associated with the Wald method are greater than or relatively close to the nominal coverage probability (0.95). Moreover, the coverage probability results for the parameters α and β associated with the profile-likelihood technique are not much stable when the sample size n is large because most of them are less than the nominal coverage probability but when the sample size n is small , most of them are relatively close to the nominal coverage probability. Using confidence length and coverage probabilities as the criteria for comparing the efficiency of the two techniques, it can be concluded that, for small samples, it would be preferable to use profile-likelihood method over the Wald method in obtaining interval estimates under type-2 censoring scheme.

iii. From the likelihood contour plots associated with both simulated and real data, it can be observed that the profile-likelihood technique provides better accuracy in jointly estimating the unknown parameters α and β of the 2-parameter Weibull distribution because even within the 90% contour likelihood region, the true population parameter of interest is still contained inside.

5.3 Recommendations

This research covers the construction of approximate confidence intervals for parameters of the 2-parameter Weibull distribution based on type-2 censored data using Profile-likelihood method. This method can be extended to constructing approximate confidence intervals for parameters of 2-parameter Weibull distribution based on hybrid censored data. Lastly, the Profile-likelihood interval estimates obtained in this research using small type-2 censored data are plausible and can be used to make better inferences in life-testing experiments especially when the effective sample size used is small. This will save time and minimize the cost of performing life-testing experiments.

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APPENDIX

Appendix I: Publication Abstract

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Interval Estimation in a Two Parameter Weibull Distribution Based on Type-2 Censored Data

Raphael Masila Mweleli, Luke Akong'o Orawo, Cox Lwaka Tamba, Justin Obwoge Okenye

Department of Mathematics, Egerton University, Egerton, Kenya
Email: raphaelrampz@yahoo.com, orawo2000@yahoo.com, clwaka@yahoo.com, justinoke69@gmail.com

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Abstract

In this paper, we consider the construction of the approximate profile- likelihood confidence intervals for parameters of the 2-parameter Weibull distribution based on small type-2 censored samples. In previous research works, the traditional Wald method has been used to construct approximate confidence intervals for the 2-parameter Weibull distribution under type-2 censoring scheme. However, the Wald technique is based on normality assumption and thus may not produce accurate interval estimates for small samples. The profile-likelihood and Wald confidence intervals are constructed for the shape and scale parameters of the 2-parameter Weibull distribution based on simulated and real type-2 censored data, and are hence compared using confidence length and coverage probability.

Keywords

Two-Parameter Weibull Distribution, Interval Estimation, Relative Likelihood Function, Maximum Relative Likelihood Function, Profile-Likelihood Interval, Coverage Probability

1. Introduction

On most occasions, when performing life testing experiments, the main interest is to examine the lifespan of a specimen. For instance, tests might be carried out to determine the lifespan of an aircraft wing prior to failure from metal fatigue. Such experiments are expensive and time-consuming, and only a few units can be inspected. Due to time and financial constraints, a researcher may not be able to examine the failure time of all the units under investigation [1] and [2]. One is prompted to set an appropriate censoring limit after which the experiment is terminated. The termination of observations in life test experiments due to

Appendix II: Research Permit

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Appendix III: R Code

```
# Simulation of Data
set.seed(271)
order341=function(alpha,beta,r,n){
  #Simulation of order statistics using the Newby Algorithm (1979)
  # This algothim is used to simulate Type II
  # Censored data from Weibull Distribution
  H=rep(NA,(r+1))
  H[1]=0
  Time1=rep(NA,(r+1))
  Time1[1]=0
  for (j in 2:(r+1)){
    U=runif(1)
    H[j]=H[j-1]-log(U)/(n-j-1)
    Time1[j]=((1/alpha)*H[j])^(1/beta)
  }
  Order=Time1[2:(r+1)]
  return(Order)
}

TT=order341(1,1.5, 8,40)
TT
### log likelikehood
loglike=function(alpha,beta,n,data){
  r=length(data)
  S=0
  for (i in 1:r){

    S=S-alpha*data[i]+(beta-1)*log(data[i])
  }
```

```
  lb=(log(factorial(n))-log(factorial(n-r))+r*(log(alpha)+log(beta))
      -(n-r)*alpha*data[r]^beta)
  l1=S+lb
  return(l1)

}

llike=function(theta){
  alpha=theta[1]; beta=theta[2]
  jk=loglike(alpha,beta,n,data)
  return(jk)

}

MLEst1=function(n,data,start,M,conf.level,true=c(1.5,1.2)){

  r=length(data)
  alpha=c()
  beta=c()
  K=1
  Xa0=start[1]
  Xb0=start[2]
  alpha[K]=start[1]
  beta[K]=start[2]

  S1f=0
  S2f=0
  S3f=0
  S4f=0
  for (i in 1:r){
```

```
  S1f=S1f+log(data[i])
  S2f=S2f+data[i]^Xb0
  S3f=S3f+(data[i]^Xb0)*log(data[i])
  S4f=S4f+(data[i]^Xb0)*(log(data[i]))^2
}

S2s=(n-r)*data[r]^Xb0
S3s=(n-r)*(data[r]^Xb0)*log(data[r])
S4s=(n-r)*(data[r]^Xb0)*(log(data[r]))^2
Xb11=(r/Xb0)+S1f-S3f-((r/(S2f+S2s))*(S3f+S3s))
Xb12=(r/Xb0^2)+S4f-((r/(S2f+S2s)^2)*(S3f+S3s)^2)+((r/(S2f+S2s))*(S4f+S4s))
Xb1=Xb0+Xb11/Xb12
#Cond=(abs(Xb1-Xb0)>=toll)
Xa1=r/(S2f+S2s)
for (m in 2:M){
  Xa1=r/(S2f+S2s)
  Xb0=Xb1
  Xa0=Xa1
  S1f=0
  S2f=0
  S3f=0
  S4f=0
  for (i in 1:r){
    S1f=S1f+log(data[i])
    S2f=S2f+data[i]^Xb0
    S3f=S3f+(data[i]^Xb0)*log(data[i])
    S4f=S4f+(data[i]^Xb0)*(log(data[i]))^2
  }

  S2s=(n-r)*data[r]^Xb0
```

```
  S3s=(n-r)*(data[r]^Xb0)*log(data[r])
  S4s=(n-r)*(data[r]^Xb0)*(log(data[r]))^2
  Xb11=(r/Xb0)+S1f-S3f-((r/(S2f+S2s))*(S3f+S3s))
  Xb12=(r/Xb0^2)+S4f-((r/(S2f+S2s)^2)*(S3f+S3s)^2)+((r/(S2f+S2s))*(S4f+S4s))
  Xb1=Xb0+Xb11/Xb12
  #Cond=(abs(Xb1-Xb0)>=toll)
  #Xa1=r/(S2f+S2s)
  K=K+1
  alpha[m]=Xa0
  beta[m]=Xb1

}
estbeta=Xb1; estalpha=Xa1
S1f1=0
S2f1=0
S3f1=0
S4f1=0
for (i in 1:r){
  S1f1=S1f1+log(data[i])
  S2f1=S2f1+data[i]^Xb1
  S3f1=S3f1+(data[i]^Xb1)*log(data[i])
  S4f1=S4f1+(data[i]^Xb1)*(log(data[i]))^2
}
S3s1=(n-r)*(data[r]^Xb1)*log(data[r])
S4s1=(n-r)*(data[r]^Xb1)*(log(data[r]))^2
I11=-r/estalpha^2
I22=-r/estbeta^2-estalpha*S4s1-estalpha*S4f1
I12=-S3s1-S3f1

INFO=solve(-1*matrix(c(I11,I12,I12,I22),2),tol=1e-60)
```

```
  WaldCIalpha=rep(NA,2)
  WaldCIbeta=rep(NA,2)
  WaldCIalpha1=estalpha-qnorm(1-conf.level/2)*(INFO[1,1])^0.5
  WaldCIalpha[2]=estalpha+qnorm(1-conf.level/2)*(INFO[1,1])^0.5
  if(WaldCIalpha1<0){WaldCIalpha[1]=0}else{WaldCIalpha[1]=WaldCIalpha1}

  WaldCIbeta1=estbeta-qnorm(1-conf.level/2)*(INFO[2,2])^0.5
  WaldCIbeta[2]=estbeta+qnorm(1-conf.level/2)*(INFO[2,2])^0.5
  if(WaldCIbeta1<0){WaldCIbeta[1]=0}else{WaldCIbeta[1]=WaldCIbeta1}

  alphaI=ifelse((WaldCIalpha[1]<true[1] & WaldCIalpha[2]>true[1]),1,0)
  betaI=ifelse((WaldCIbeta[1]<true[2] & WaldCIbeta[2]>true[2]),1,0)

  return(list(K=K,beta=beta[1:K],alpha=alpha[1:K],estalpha=estalpha,
        estbeta=estbeta,INFO=INFO,WaldCIalpha=WaldCIalpha,
        WaldCIbeta=WaldCIbeta,alphaI=alphaI,betaI=betaI))

}



K=8
N=40

Samples=matrix(rep(NA,K*1000),nrow = 1000)
EstMs=matrix(rep(NA,2*1000),nrow = 1000)

VarEst=matrix(rep(NA,2*1000),nrow = 1000)
```

```
for (j in 1:1000){
  #WWW=rweibull(N,0.5,1.5)
  #Samples[j,]=sort(WWW)[1:K]  #order341(1.5,1.5,K,N)
  Samples[j,]=order341(1,1.5,K,N)
}
```

#Wald Coverage Probability

```
loglike1=function(par,n,data){
  alpha=par[1]
  beta=par[2]
  r=length(data)
  S=0
  for (i in 1:r){

    S=S-alpha*data[i]+(beta-1)*log(data[i])
  }
  lb=(log(factorial(n))-log(factorial(n-r))+r*(log(alpha)+log(beta))
      -(n-r)*alpha*data[r]^beta)
  l1=S+lb
  return(l1)

}
loglike1(c(1.5,1.2),N,Samples[500,])

aa=function(par){
  data=Samples[500,]
  n=N
  aa1=-loglike1(par,n,data)
  return(aa1)

}
```

```
WaldOP=matrix(rep(NA,1000*8),nrow = 1000)
for (i in 1:1000){
  if(i%%100==0){print(i)}
  true=c(1,1.5)
  dat1=Samples[i,]
  aa=function(par){
    data=dat1
    n=N
    aa1=-loglike1(par,n,data)
    return(aa1)

  }

  dff=optim(c(.1,.2),aa,method = 'L-BFGS-B',
          lower = 10^-10,control = list(maxit=5000),hessian = TRUE)
  WaldOP[i,1:2]=dff$par
  INFO=solve(dff$hessian)
  conf.level=.05

  Wa1=dff$par[1]-qnorm(1-conf.level/2)*(INFO[1,1])^0.5
  Wa2=dff$par[1]+qnorm(1-conf.level/2)*(INFO[1,1])^0.5
  if(Wa1<0){WaldOP[i,3]=0}else{WaldOP[i,3]=Wa1}
  if(Wa1<0){Wa11=0}else{Wa11=Wa1}
  WaldOP[i,4]=Wa2

  Wb1=dff$par[2]-qnorm(1-conf.level/2)*(INFO[2,2])^0.5
  Wb2=dff$par[2]+qnorm(1-conf.level/2)*(INFO[2,2])^0.5
  if(Wb1<0){WaldOP[i,5]=0}else{WaldOP[i,5]=Wb1}
  WaldOP[i,6]=Wb2
```

```
  alphaI=ifelse((Wa1<true[1] & Wa2>true[1]),1,0)
  betaI=ifelse((Wb1<true[2] & Wb2>true[2]),1,0)
  WaldOP[i,7]=alphaI
  WaldOP[i,8]=betaI

}
WaldOP1=as.data.frame(WaldOP)
names(WaldOP1)=c('alphaE','betaE','LP_alpha','UP_alpha','LP_beta','UP_beta','alphaP_I','betaP_I')
View(WaldOP1)
cpP_a1=sum(WaldOP1$alphaP_I)/1000
cpP_b1=sum(WaldOP1$betaP_I)/1000
cpP_a1
cpP_b1

#Profile-Likelihood

lRL=function(alpha,beta,alphahat,betahat,n,data){
  RR=loglike(alpha,beta,n,data)-loglike(alphahat,betahat,n,data)
  return(RR)

}

lRL(1,1.5,RR$alpha[1000],RR$beta[1000],N,Samples[500,])

exp(lRL(1,1.5,RR$alpha[1000],RR$beta[1000],N,Samples[500,]))

gg=function(beta){
  lRL(0.3536448,beta,0.3536448,1.0860346,N,Samples[500,])
}
```

```
loglike1=function(par,n,data){
  alpha=par[1]
  beta=par[2]
  r=length(data)
  S=0
  for (i in 1:r){

    S=S-alpha*data[i]+(beta-1)*log(data[i])
  }
  lb=(log(factorial(n))-log(factorial(n-r))+r*(log(alpha)+log(beta))
      -(n-r)*alpha*data[r]^beta)
  l1=S+lb
  return(l1)

}
loglike1(c(1.5,1.2),N,Samples[500,])

aa=function(par){
  data=Samples[500,]
  n=N
  aa1=-loglike1(par,n,data)
  return(aa1)

}

dff=optim(c(.1,.2),aa,control = list(maxit=5000),hessian=TRUE)
```

```
dff

#######
alpha=seq(.1,5,.01)
beta=seq(.1,5,.01)
MM=length(alpha)
LLL=matrix(rep(NA,MM*MM),MM)
for (i in 1:MM){
  for (j in 1:MM){
    LLL[i,j]=loglike(alpha[i],beta[j],n=N,Samples[500,])
  }
}

contour(alpha, beta, LLL, zlim = c(0, 7),nlev = 20, lty = 1, method = "simple",
        main = '',xlab=expression(alpha),ylab=expression(beta))

abline(h=c(2,4), v=c(1,4),col='blue')
abline(h=c(2.5,3.5), v=c(1.5,3.5),col='red')

qw=which(LLL==max(LLL),arr.ind = T)
cc<-c(alpha[qw[1]],beta[qw[2]])



gg=function(beta){
  exp(lRL(dff$par[1],beta,dff$par[1],dff$par[2],N,Samples[500,]))
}
plot(gg,0,5)
abline(h=1,col='blue')
abline(h=.147,col='red')
gg2=function(alpha){
```

```
  exp(lRL(alpha,dff$par[2],dff$par[1],dff$par[2],N,Samples[500,]))
}
plot(gg2,0,5)
abline(h=.147,col='red')

gg1=function(beta){gg(beta)-0.147}
plot(gg,0,5,xlab=expression(beta),ylab=expression(R[P](beta)))
abline(h=0.147,col='red')

c(uniroot(gg1,c(0,dff$par[2]))$root,uniroot(gg1,c(dff$par[2],5))$root)
aaa=c(uniroot(gg1,c(0,dff$par[2]))$root,uniroot(gg1,c(dff$par[2],5))$root)
aaa
abline(v=aaa,col='green')
abline(v=c( Wb1, Wb2),col='blue')

diff(c(uniroot(gg1,c(0,dff$par[2]))$root,uniroot(gg1,c(dff$par[2],5))$root))
gg3=function(alpha){gg2(alpha)-.147}
plot(gg2,-1,5,xlab=expression(alpha),ylab=expression(R[P](alpha)))
abline(h=0.147,col='red')

c(uniroot(gg3,c(0,dff$par[1]))$root,uniroot(gg3,c(dff$par[1],5))$root)
diff(c(uniroot(gg3,c(0,dff$par[1]))$root,uniroot(gg3,c(dff$par[1],5))$root))
bbb=c(uniroot(gg3,c(0,dff$par[1]))$root,uniroot(gg3,c(dff$par[1],5))$root)
bbb
abline(v=bbb,col='green')
abline(v=c(Wa1, Wa2),col='blue')
```

#Profile-likelihood coverage probability code

```
lRL=function(alpha,beta,alphahat,betahat,n,data){
```

```
  RR=loglike(alpha,beta,n,data)-loglike(alphahat,betahat,n,data)
  return(RR)

}

lRL(0.5,1.5,RR$alpha[1000],RR$beta[1000],N,Samples[500,])

exp(lRL(0.5,1.5,RR$alpha[1000],RR$beta[1000],N,Samples[500,]))

gg=function(beta){
  lRL(0.5569005,beta,0.5569005,1.0555771,N,Samples[500,])
}

loglike1=function(par,n,data){
  alpha=par[1]
  beta=par[2]
  r=length(data)
  S=0
  for (i in 1:r){

    S=S-alpha*data[i]+(beta-1)*log(data[i])
  }
  lb=(log(factorial(n))-log(factorial(n-r))+r*(log(alpha)+log(beta))
      -(n-r)*alpha*data[r]^beta)
  l1=S+lb
  return(l1)

}
loglike1(c(1.5,1.2),N,Samples[500,])
```

```
aa=function(par){
  data=Samples[500,]
  n=N
  aa1=-loglike1(par,n,data)
  return(aa1)

}

dff=optim(c(.1,.2),aa,control = list(maxit=5000),hessian = TRUE)



#######
alpha=seq(.00000001,15,.01)
beta=seq(.00000001,15,.01)
MM=length(alpha)
LLL=matrix(rep(NA,MM*MM),MM)
for (i in 1:MM){
  for (j in 1:MM){
    LLL[i,j]=loglike(alpha[i],beta[j],n=N,Samples[59,])
  }
}

contour(alpha, beta, LLL, xlim=c(0,1),ylim=c(0,13),zlim = c(0, 40),nlev = 30, lty = 2,
method = "simple",
        main = "Contour Plot")



qw=which(LLL==max(LLL),arr.ind = T)
cc<-c(alpha[qw[1]],beta[qw[2]])
```

```
gg=function(beta){
  exp(lRL(cc[1],beta,cc[1],cc[2],N,Samples[39,]))
}
plot(gg,0,5)
abline(h=1,col='blue')
abline(h=.147,col='red')
gg2=function(alpha){
  exp(lRL(alpha,dff$par[2],dff$par[1],dff$par[2],N,Samples[500,]))
}
plot(gg2,0,5)
abline(h=.147,col='red')

gg1=function(beta){gg(beta)-.147}
plot(gg1,0,5)
abline(h=0,col='red')

c(uniroot(gg1,c(0,dff$par[2]))$root,uniroot(gg1,c(dff$par[2],100))$root)

diff(c(uniroot(gg1,c(0,1))$root,uniroot(gg1,c(2,5))$root))
gg3=function(alpha){gg2(alpha)-.147}
plot(gg3,0,5)
abline(h=0,col='red')

c(uniroot(gg3,c(0,dff$par[1]))$root,uniroot(gg3,c(dff$par[1],3))$root)
diff(c(uniroot(gg3,c(0,0.7))$root,uniroot(gg3,c(0.7,3))$root))

Profiles=matrix(rep(NA,1000*6),nrow = 1000)

for (i in 1:1000){
  true=c(0.5,1.5)
```

```
  dat1=Samples[i,]
  aa=function(par){
    data=dat1
    n=N
    aa1=-loglike1(par,n,data)
    return(aa1)

  }

  dff=optim(c(.1,.2),aa,method = 'L-BFGS-B',
            lower = 10^-10,control = list(maxit=5000),hessian = TRUE)
  gg=function(beta){
    exp(lRL(dff$par[1],beta,dff$par[1],dff$par[2],N,dat1))
  }
  gg1=function(beta){gg(beta)-.147}
  bc=c(uniroot(gg1,c(0,dff$par[2]))$root,uniroot(gg1,c(dff$par[2],100))$root)
  gg2=function(alpha){
    exp(lRL(alpha,dff$par[2],dff$par[1],dff$par[2],N,dat1))
  }
  gg3=function(alpha){gg2(alpha)-.147}
  ac=c(uniroot(gg3,c(0,dff$par[1]))$root,uniroot(gg3,c(dff$par[1],100))$root)
  Profiles[i,1:4]=c(ac,bc)
  Profiles[i,5]=ifelse((ac[1]<true[1] & ac[2]>true[1]),1,0)
  Profiles[i,6]=ifelse((bc[1]<true[2] & bc[2]>true[2]),1,0)

}

Profiles1=as.data.frame(Profiles)
names(Profiles1)=c('LP_alpha','UP_alpha','LP_beta','UP_beta','alphaP_I','betaP_I')
View(Profiles1)
```

```
cpP_a=sum(Profiles1$alphaP_I)/1000
cpP_b=sum(Profiles1$betaP_I)/1000
cpP_a
cpP_b
```

############ profile-Likelihood contour R code

```
N1=500
true=c(1,1.5)
LLL=function(par){exp(loglike1(par,N,Samples[N1,]))}
C=LLL(c(WaldOP1[N1,1],WaldOP1[N1,2]))
R_LL=function(a,b){par=c(a,b)
(1/C)*LLL(par)}
a2=seq(.01,10,length=1000)

b2=seq(.01,5,length=1000)

#z=outer(a2,b2,R_LL)
z=matrix(rep(NA,length(a2)*length(b2)),length(b2))
for(i in 1:length(a2)){
  for(j in 1:length(b2)){
    z[i,j]=R_LL(a2[i],b2[j])
  }
}

contour(a2,b2,z,levels=c(.147,.5,.75),xlab='alpha',ylab='beta')

x=WaldOP1[N1,1];y=WaldOP1[N1,2]
DD=cbind(x,y)
XX=cbind(true[1],true[2])
```

```
plot(DD,pch=8,cex=1,xlim = c(0,4),ylim = c(0,2.5),xlab='alpha',ylab='beta')
points(XX,col='red',pch=8,cex=1)
contour(a2,b2,z,levels=c(.036,.147,0.258),
        col=c('black','black','black'),xlab='alpha',ylab='beta'
        , add = TRUE)
```

#Real Data

```
data2=c(0.575,0.778,0.88,0.984,1.021,1.053,1.393,1.439,1.480)
data2
TT=data2
TT
```

log likelikehood

```
loglike=function(alpha,beta,n,data){
  r=length(data)
  S=0
  for (i in 1:r){

    S=S-alpha*data[i]+(beta-1)*log(data[i])
  }
  lb=(log(factorial(n))-log(factorial(n-r))+r*(log(alpha)+log(beta))
      -(n-r)*alpha*data[r]^beta)
  l1=S+lb
  return(l1)

}
```

#Profile-Likelihood

```
lRL=function(alpha,beta,alphahat,betahat,n,data){
  RR=loglike(alpha,beta,n,data)-loglike(alphahat,betahat,n,data)
```

```
  return(RR)

}

loglike1=function(par,n,data){
  alpha=par[1]
  beta=par[2]
  r=length(data)
  S=0
  for (i in 1:r){

    S=S-alpha*data[i]+(beta-1)*log(data[i])
  }
  lb=(log(factorial(n))-log(factorial(n-r))+r*(log(alpha)+log(beta))
      -(n-r)*alpha*data[r]^beta)
  l1=S+lb
  return(l1)

}
```

#Wald Real

```
loglike1(c(1.5,1.2),12,data2)

aa=function(par){
  data=data2
  n=12
  aa1=-loglike1(par,n,data)
  return(aa1)

}
```

```
dff1=optim(c(1,2),aa,control = list(maxit=5000),hessian=TRUE)

dff1
WaldOP=dff1$par
INFO1=solve(dff1$hessian)
conf.level=.05

Wa1=dff1$par[1]-qnorm(1-conf.level/2)*(INFO1[1,1])^0.5
Wa2=dff1$par[1]+qnorm(1-conf.level/2)*(INFO1[1,1])^0.5
c(Wa1,Wa2)
Q=c(Wa1,Wa2)
Q
diff(c(Wa1,Wa2))

Wb1=dff1$par[2]-qnorm(1-conf.level/2)*(INFO1[2,2])^0.5
Wb2=dff1$par[2]+qnorm(1-conf.level/2)*(INFO1[2,2])^0.5
K=c(Wb1,Wb2)
K
diff(c(Wb1,Wb2))
#######
alpha=seq(.1,1.2,.01)
beta=seq(.1,10,.01)
MM=length(alpha)
NN=length(beta)
LLL=matrix(rep(NA,MM*NN),MM)
for (i in 1:MM){
  for (j in 1:NN){
    LLL[i,j]=loglike(alpha[i],beta[j],n=12,data2)
  }
```

```
}

contour(alpha, beta, LLL, zlim = c(0, 30),nlev = 30, lty = 2, method = "simple",
        main = "Contour Plot",xlab=expression(alpha),
        ylab=expression(beta))
```

#Profile Real Data

```
gg=function(beta){
  exp(lRL(dff1$par[1],beta,dff1$par[1],dff1$par[2],12,data2))
}
plot(gg,0,10)
abline(h=1,col='blue')
abline(h=.147,col='red')
gg2=function(alpha){
  exp(lRL(alpha,dff1$par[2],dff1$par[1],dff1$par[2],12,data2))
}
plot(gg2,0,1.2)
abline(h=c(0.147,1),col=c('red','blue'))
```

####MRLF OF BETA

```
gg1=function(beta){gg(beta)-0.147}
bb=c(uniroot(gg1,c(0,dff1$par[2]))$root,uniroot(gg1,c(dff1$par[2],10))$root)
bb
par(mfrow=c(1,2))
plot(gg,0,10,main='Profile-likelihood
Method',xlab=expression(beta),ylab=expression(R[P](beta)))
segments(bb[1],0.147,bb[2],.147,col='red')
plot(gg,0,10,main='Wald Method',xlab=expression(beta),ylab=expression(R[P](beta)))
segments(K[1],0.147,K[2],0.147,col='red')
##########################################
#abline(h=0.147,col='red')
```

```
#abline(v=bb,col='green')

#abline(v=c(Wb1,Wb2),col='blue')

diff(c(uniroot(gg1,c(0,dff1$par[2]))$root,uniroot(gg1,c(dff1$par[2],10))$root))
```

######MRLF OF ALPHA

```
gg3=function(alpha){gg2(alpha)-.147}

par(mfrow=c(1,2))

plot(gg2,0,1.2,main='Profile-likelihood Method',xlab=expression(alpha),ylab=expression(R[P](alpha)))

aaa=c(uniroot(gg3,c(0,dff1$par[1]))$root,uniroot(gg3,c(dff1$par[1],5))$root)

aaa

segments(aaa[1],0.147,aaa[2],0.147,col='red')

plot(gg2,0,1.2,main='Wald Method' ,xlab=expression(alpha),ylab=expression(R[P](alpha)))

segments(Q[1],0.147,Q[2],0.147,col='red')

##############

abline(v=aaa,col='green')

abline(v=c(Wa1,Wa2),col='blue')

diff(c(uniroot(gg3,c(0,dff1$par[1]))$root,uniroot(gg3,c(dff1$par[1],5))$root))
```