THE EFFECTS OF REYNOLDS NUMBER ON FLUID FLOWS

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BY

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A Dissertation submitted to graduate school, in partial fulfillment of the requirements of the Degree of Master of Science in Applied Mathematics in the Department of Mathematics Egerton University.

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RECOMMENDATION

This document was submitted for examination with my approval as supervisor.

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DEDICATION

This dissertation is dedicated to my parents, Mr. Philip Chepkwony Cheboskwony and Mrs. Jeniffer Targok Chepkwony. It is also dedicated to my wife Josphine Jemutai Kurui and my daughter Viola Jeruto Kurui for their prayers, financial support and perseverance during my long study period.

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ABSTRACT

This dissertation investigates the effects of Reynolds number (*Re*) to fluid flows. Very small , moderate and large Reynolds number effects are considered and their corresponding significance are discussed. For small Reynolds number, creeping flows are considered, so that the Navier-Stokes equations and the continuity equation for creeping flow are derived. These two equations are further reduced to Laplace equation, which can easily be solved. This theory is therefore applied to solve problems related to very slow motion flows e.g. flow through porous media, the Helle Shaw flow, the flow past a sphere and the lubrication theory. However, emphasis has been placed on flow through porous media.

For moderate Reynolds numbers, laminar flows and flow through circular pipes; minor and major head losses in pipe flows are also considered. Further, transitions from laminar to turbulent flow regimes in pipe flows are discussed with reference to the Reynolds number.

For the large Reynolds numbers flows, the boundary layer concept is discussed where the boundary layer thickness is calculated. The skin friction coefficient, the drag coefficient, and lift coefficient acting on objects moving within the boundary layer are also considered, in these cases, discussions are made on how these coefficients can be increased or minimised. Finally boundary layer separation and control are discussed and their relevant applications are considered with respect to motion of objects influenced by the existence of boundary layer flows.

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NOMENCLATURE

$\vec{F}$	The conservative force field associated with fluid flow.
ψ	Stream function
t	Time
p	pressure of the fluid
∇	The operator *del.* equivalent to $\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$
g	Force due to gravity
w	The complex potential
k	A constant
Fr	Froude number
Ma	Mach number
$\frac{e}{l}$	Relative roughness of a pipe
Re	Reynolds number
ρ	Fluid density
δ	Boundary Layer thickness
δ_1	Displacement thickness
δ_2	Momentum thickness
δ_3	Energy thickness
Re_x	Reynolds number at a distance x from the leading edge of a plate
c_f	Skin- friction coefficient

C_f Plate skin- friction coefficient

D Drag force

V Arbitrary fluid volume

v fluid velocity

dv volume element

μ coefficient of viscosity

ϑ viscosity

S Bounding surface of a fluid

ds Surface element of a fluid

$\hat{n}$ Unit normal to a surface

$\delta\xi$ Fluid displacement

$\bar{T}$ Surface Traction force

s Porosity of a material

Ω Gravitational potential

Φ velocity potential

q Fluid flux

C_D Drag coefficient

C_L Lift coefficient

L Lift Force

l Characteristic length

α Angle of attack of an airfoil

τ Shear stress

CHAPTER ONE

BASIC EQUATIONS GOVERNING A FLUID FLOW

1.1 Introduction

This dissertation finds out the effects of Reynolds number on fluid flows. Both large, small and very small Reynolds number effects are to be considered. The fluid is assumed to be incompressible, viscous and irrotational. In each case, the effects of Reynolds number are considered and their respective applications or uses are also discussed. A very important aspect of Reynolds number is that it gives the ratio of inertia forces to viscous forces in the fluid motion, and this affects the flow patterns in one way or another.

1.2 The Governing Equations

1.2.1 The Reynold's Number Equation

The inertia force (i.e. product of mass and acceleration) always exist in all fluid flow Problems. Besides the inertia force, there always exist some additional forces which are responsible for fluid motion. The required condition for dynamic similarity can always be obtained by considering the ratio of the inertia force to any one of the remaining forces e.g. gravity force, viscous force, pressure force, elastic force and so on. The ratios of forces gives some dimensional numbers, in which Reynolds number is one of them.

Reynold's number is defined as the ratio of inertia force to the visous force.

$$\text{i.e } \mathrm{Re} = \frac{Inertia\ force}{Viscous\ force} \tag{1}$$

$$= \frac{Mass\ x\ accelation}{shear\ stress\ x\ cross\ sectional\ area}$$

$$= \frac{volume\ x\ Density\ x\ (velocity / time}{shear\ stress\ x\ cross-sectional\ area}$$

$$= \frac{cross\ sectional\ area\ x\ linear\ dimension\ x\ density\ x \frac{velocity}{time}}{shear\ stress\ x\ cross-sectional\ area}$$

$$= \frac{(Velocity)^2\ x\ density}{shear-stress}$$

$$= \frac{V}{\mu\left(\frac{\partial u}{\partial y}\right)}$$

$$= \frac{V^2 \rho}{\mu\left(\frac{V}{L}\right)}$$

$$\Rightarrow \mathrm{Re} = \frac{VL\rho}{\mu} \tag{2}$$

or

$$\mathrm{Re} = \frac{VL}{\nu} \tag{3}$$

where $\nu = \mu/\rho$

L = *characteristic length*

V = *characteristic velocity*

Re = Reynolds number

Hence from Equation (1) it follows that when *Re* is small, the viscous force is predominant and the effect of viscosity is important in the whole velocity field. When *Re* is large then the inertia force is predominant and the effect of viscosity is less important, and consequently the fluid may be treated as non-viscous.

For geometric similarity of two flows, it is necessary that their Reynolds number should be the same and the boundary conditions are satisfied.

1.2.2 Derivation of Momentum Equations (Navier-Stokes Equation)

In the motion of any particle, we have the rate of change of momentum of the particle being equal to the impressed force. (i.e. Newton's second law of motion). We apply this principle to the motion of the fluid . In any portion of the fluid, the rate of change of momentum must be equal to the total force acting on it.

Taking an arbitrary Volume V bounded by a smooth oriental surface S wholly immersed in the fluid and inside the volume, lets take a volume element dv around a point P where the fluid density is ρ the velocity is $\vec{V}(u,v,w)$ and body force $\vec{F}(F_x,F_y,F_z)$

Then, the momentum of this fluid element along x-axis is

Mass x Velocity = $\rho dv.u$

$$= \rho u dv \tag{4}$$

Thus the rate of change of increase of momentum of the whole fluid in x-direction is

$$\frac{\hat{C}}{\hat{C}t}\int_v \rho u dv \tag{5}$$

Now we take that the fluid is leaving the volume V through each point of its surface and therefore it will take away some momentum along with it. Thus if we consider a surface element ds the bounding surface S, then the fluid flux through this cross-section is $\rho \hat{C} s \delta\varepsilon$, where $\delta\varepsilon$ is the displacement of the fluid through ds in time dt. Consequently the flux of momentum along x-axis through the surface element ds is $\rho \hat{C} s \delta\varepsilon$ .Hence the flux of Momentum along x-axis through the whole surface S is

$$\int_s \rho u \vec{v} \hat{n} ds \tag{6}$$

where $\hat{n}$ is a unit normal to the surface element ds

The body force acting on the fluid along x-axis is given by

$$\int_v \rho \vec{F}_x dv \tag{7}$$

Further, since stressess will also be acting on the fluid, then we also consider their contribution towards the total force.

Let $T_x \hat{n} dx$ be the surface traction acting along x-axis on the surface element ds of the surface S with Unit normal $\hat{n}$. Then the total surface force on the fluid along x-axis becomes

$$\int_s T_x \hat{n} ds \tag{8}$$

and by Gauss's theorem, Equation (8) becomes

$$\int_v \nabla . \vec{T}_x ds \tag{9}$$

Finally, from principle of conservation of Momentum, the rate of increase of the whole Momentum of the fluid is given by

$$\frac{\partial}{\partial t} \int_v \rho u dv = -\int_s \rho u \vec{v} \hat{n} ds + \int_v \nabla . \vec{T}_x dv + \int_v \rho \vec{F}_x dv \tag{10}$$

Since the volume v has been taken to be fixed in the fluid, then on the left hand side of Equation (10) the operation of integration and partial derivative with respect to time can be interchanged.

Thus using Gauss's theorem, in the first integral of the R.H.S. of (10), we get

$$\int_v \left[\frac{\partial}{\partial t}(\rho u) + \nabla . (\rho u \vec{v}) - \nabla \vec{T}_x - \rho \vec{F}_x \right] dv = 0 \tag{11}$$

Since we have taken the volume V to be arbitrary, then the integration in equation (11) should be zero.

Thus $\frac{\hat{C}}{\hat{C}t}(\rho u)+\nabla.(\rho u\vec{v})-\nabla\vec{T}_x-\rho\vec{F}_x=0$

Implying that

$$\rho\frac{\hat{C}u}{\hat{C}t}+\rho\vec{v}\nabla.u=\nabla\vec{T}_x+\rho\vec{F}_x \tag{12}$$

Similarly along y *and* z axes we shall have

$$\rho\frac{\hat{C}v}{\hat{C}t}+\rho\vec{v}\nabla.v=\nabla\vec{T}_y+\rho\vec{F}_y \tag{13}$$

$$\rho\frac{\hat{C}w}{\hat{C}t}+\rho\vec{v}\nabla.w=\nabla\vec{T}_z+\rho\vec{F}_z \tag{14}$$

Equations (12) - (14) can be written in Vector form as

$$\rho\frac{\hat{C}\vec{v}}{\hat{C}t}+\rho\vec{v}.\nabla.\vec{v}=\nabla\vec{T}+\rho\vec{F} \tag{15}$$

which is the Navier-stokes Equation of motion.

If the fluid is incompressible and the coefficient of viscosity μ is constant, Equation (15) becomes

$$\rho\frac{\hat{C}\vec{v}}{\hat{C}t}+\rho\vec{v}.\nabla.\vec{v}=\rho\vec{F}-\nabla p+\mu\nabla^2\vec{v} \tag{16}$$

where

$-\nabla p+\mu\nabla^2\vec{v}=\nabla\vec{T}$ (Traction force)

$-\nabla p$ = Pressure force

$\mu\nabla^2\vec{v}$ = Frictional force

If the fluid has no external force, Equation (16) reduces to

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v}.\nabla.\vec{v} = -\nabla p + \mu \nabla^2 \vec{v} \qquad (17)$$

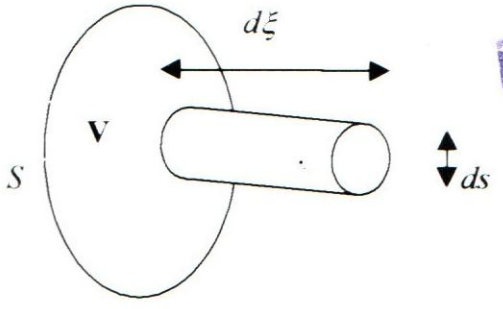
which is the momentum equation for a viscous fluid.

1.2.3 Derivation Continuity Equation (Equation of Conservation of Mass)

Consider a fluid with arbitrary Volume V fixed and bounded by a smooth oriental surface S. Let dv be volume element at point P in volume V and ρ the density at that point.

Then total mass (M) of fluid in volume V is

$$M = \int_v \rho \, dv \qquad (18)$$



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Figure 1.1 Arbitrary Volume of a fluid

Taking the surface element ds whose normal normal is $\hat{n}$ and let in time δt the fluid passing through ds be displaced by $d\varepsilon$ and thus a fluid element of mass (M) = $\rho ds d\varepsilon$ moves out through the surface element ds

The rate at which the fluid is leaving the volume through ds is

$$\lim_{\delta t \to o} \frac{\rho ds d\varepsilon}{dt} = \rho \vec{v} \hat{n} ds \tag{19}$$

where $\vec{v}$ is the fluid velocity at the surface element. Thus the outward flux of the fluid through the whole surface S is given by

$$\int_s \rho \vec{v} \hat{n} ds \tag{20}$$

According to the principle of conservation of mass, the rate of increase of the total mass of fluid in volume V must be equal to the negative of the outward flux of the fluid through surface S

$$\text{that's} \quad \frac{\hat{c}}{\hat{c}t} \int_v \rho dv = -\int_s \rho \vec{v} \hat{n} ds \tag{21}$$

Since the volume considered is fixed, then integration and time derivative operators can be interchanged, moreover applying Gauss's theorem on integration in the right hand side of the equation (21) yields

$$\int \left[\frac{\hat{c} \rho}{\hat{c}t} + \nabla (\rho \vec{v}) \right] dv = o \tag{22}$$

because volume V is an arbitrary constant, the integral varnishes and we remain only with

$$\frac{\hat{c} \rho}{\hat{c}t} + \nabla . \rho \vec{v} = o \tag{23}$$

and hence for steady incompressible fluid flow, equation (23) reduces to

$$\nabla . \vec{v} = o \tag{24}$$

which is the continuity equation.

In cartesian coordinate system equation (24) becomes

$$\frac{\hat{c}u}{\hat{c}x} + \frac{\hat{c}v}{\hat{c}y} + \frac{\hat{c}w}{\hat{c}z} = o \tag{25}$$

and in cylindrical coordinate system, it becomes

$$\frac{\partial u_r}{\partial r} + \frac{U_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = o \qquad (26$$

1.2.4 Velocity Potential and Stream Function Equation.

The two-dimensional incompressible continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (27)$$

guarantees the existence of a stream function ψ, from which the velocity components can be derived as

$$u = \frac{\partial \psi}{\partial y} \qquad , v = \frac{-\partial \psi}{\partial x} \qquad (28)$$

likewise, the condition of irrotationality

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (29)$$

guarantees the existence of another scalar function ϕ, called the velocity potential, which is related to the velocity components by

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} \qquad (30)$$

since a velocity potential must exist in all irrotational flows, such flows are frequently called *potential flows*. Equations (28) and (30) imply that the derivative of ψ gives the velocity component in a direction 90° clockwise from the direction of differentiation, whereas the derivative of ϕ gives the velocity components in the direction of differentiation. Comparing equation (28) and (30), it follows that

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y}\\ \frac{\partial\phi}{\partial y} &= \frac{-\partial\psi}{\partial x}\end{aligned}\right\} \tag{31}$$

called the Cauchy-Rieman conditions from which one of the functions can be determined if the other is known.

Stream function and velocity potential satisfy the Laplace equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{32}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{33}$$

as can be seen by cross differentiation of equation (31)

Equation (33) holds for two - dimensional flow only, because stream function cannot be defined in a completely three- dimensional flow. However, velocity potential ϕ, can be defined in three-dimensional irrotational flows, because $u = \nabla\phi$ identically satisfies the irrotationality condition $\vec{\nabla} \times \vec{u} = 0$.

1.3 Literature Review

This dissertation investigates the effects of Reynolds number on fluid flows. That is how Reynolds number affects the particular kind of flow and the significance of such effects. Unlike many literature in books, this dissertation brings the specific effects and its significance directly.

There are many physical situations where Reynolds number effects are experienced and employed. Bruce R.M and et.al (1994) addressed these Reynolds number broadly when discussing Reynolds number as one of the dimensionless parameters. They pointed out that Reynolds number represents a combination of fluid variables that could be used as a criterion for distinguishing between Laminar and Turbulent flow. They further identified Reynolds number as the most important dimensionless parameter in a pipe flow.

Pnueli and Gutfinger (1992) in discussing similitude and order of magnitude discussed the significance of the Reynolds number in similarity of two flows or similarity condition for two vessels or objects under study. Their work pointed out that Reynolds number is important in obtaining knowledge of an actual flow after studying the prototype flow.

Askel L. L. (1979) did also some work on the Reynolds number effects on fluid flow. His work explained the significance of Reynolds number in calculation of heat transfer coefficients such as the Grashof number and the Nusselt number. These coefficients are very important in studying forced and free convection in heat transfers in an heat exchanger tubes. Similarly the work of F. Chorton(1985) while discussing the Prandtl boundary layer theory outlines the Reynolds number effects and its significance in the boundary layer region. He points out that the Reynolds number ensures dynamic

similarity at corresponding points near the boundaries where viscous effects supervene. He further discusses the importance of the Reynolds number effects in simulating or modelling a given flow. He says that "it is seldom, if ever, possible to keep Reynolds number constant and preserve geometric similarity".

The availability in recent times of high performance computers has inspired the development of more Reynolds number effects and applications in complex fluid flow fields.

1.4 Dissertation Outline

As aforementioned, this project investigates the effects of Reynolds number on fluid flow. Not all effects are considered, but a few amongst many effects have been considered.

In chapter one, we have introduced in general sense the project, provided the governing equations and a brief literature review of the same.

In chapter two, we considered the small Reynolds number effects, that's the creeping flows and in particular the flows through a porous media.

Chapter three is devoted to laminar and turbulent boundary layer. In this chapter we considered situations where the large Reynolds number effects are encountered and their significance.

For chapter four, Laminar and turbulent flows in circular pipes are considered. The effects of Reynolds number in pipe fow problems is looked into and its significance in the calculation of pipe head losses is discussed.

In chapter five, the applications of effects of large Reynolds number is further dealt with. The drag force, the lift force, and the boundary layer separation and control are discussed.

CHAPTER TWO

SMALL REYNOLDS NUMBER EFFECTS AND ITS APPLICATIONS

2.1 Introduction

Navier – Stokes equations are non- linear equations, as previously derived, see equation (17). The solution to these equations in a general case is not simple. The main difficulty in obtaining a solution is the presence of non – linear terms. One of the situations where these terms are not important is when the Reynolds number Re is very small.

Small Reynolds number flows are roughly categorized into two groups;

Definition 1; The Creeping Flows

These are the flows when the velocity and dimensions of the body is small, such that the Reynolds number is very much less than one ($Re << 1$). However ,the solutions obtained for this motion hold when the Reynolds number is simply less than one or even greater than one , but has a moderate value.

Definition 2; The Viscous Flows

When the viscosity of the fluid is very high and the dimensions of the body are very small for example the motion of oils used as lubricants.

2.2 Creeping Flows

Since in these small Reynolds number flows the non – linear terms are unimportant we simply neglect them and for simplicity we take the fluid to be incompressible and hence form the relevant equations governing their flow.

2.2.1 Mathematical Formulation

The Navier – Stokes equation and the mass conservation equation for incompressible fluid are of the form

$$\text{div. } \vec{v} = 0 \tag{34}$$

$$\text{and grad. } P = \mu \nabla^2 \vec{v} \tag{35}$$

or in extended form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{36}$$

and

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \\ \frac{\partial p}{\partial y} &= \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \\ \frac{\partial p}{\partial z} &= \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] \end{aligned}\right\} \tag{37}$$

respectively.

The equations (36) and (37) must be supplemented with the same boundary conditions as of the full Navier-stokes equations. That is the normal and tangential components of velocity should be continuous at the boundary or in other words those conditions expressing the absence of the slip in the fluid at the wall.

Differentiating equations (37) with respect to x, y, and z respectively and adding them we obtain

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=\mu\left[\begin{array}{l}\dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}\right)+\dfrac{\partial^2}{\partial y^2}\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}\right)+ \\ \dfrac{\partial^2}{\partial z^2}\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}\right)\end{array}\right] \tag{38}$$

Substituting equation (36) into equation (38) gives

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial z^2}=0$$

which implies that $\nabla^2 p=0$ (39)

equation (39) shows that the pressure $p(x,y,z)$ satisfies the Laplace's equation which is an important result in analyzing small Reynolds number flows. It shows that the pressure field $p(x,y,z)$ in a creeping motion satisfies the potential and thus the pressure $p(x,y,z)$ is a *potential function*. This result shows that the pressure at any point in the flow can be determined independently and depends only on the pressure at the boundary as a solution of Laplace's equation is uniquely determined in terms of the value of the variable at the boundary.

Consequently ,the pressure at any point is independent of viscosity and the density of the fluid. To obtain the velocity components we solve equations (36) and (37)

simultaneously. We introduce a stream function (ψ) and convert these equations into one equation of higher order and then solve resultant equation.

We know that the stream function (ψ) is defined by the Cauchy-Rieman equations as

$$u = \frac{\partial \psi}{\partial y}$$

and (40)

$$v = \frac{-\partial \psi}{\partial x}$$

Eliminating pressure (p) from the first two equations of (37) and introducing stream function (ψ) we obtain

$$\frac{\partial p}{\partial x} = \mu \left[\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3} + \frac{\partial^3 \psi}{\partial z^2 \partial y} \right] \tag{41}$$

$$\frac{\partial p}{\partial y} = \mu \left[-\frac{\partial^3 \psi}{\partial y^2 \partial x} - \frac{\partial^3 \psi}{\partial x^3} - \frac{\partial^3 \psi}{\partial z^2 \partial x} \right] \tag{42}$$

Differentiating equation (41) and (42) with respect to y and x respectively we obtain

$$\frac{\partial^2 p}{\partial x \partial y} = \mu \left[\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} + \frac{\partial^4 \psi}{\partial z^2 \partial y^2} \right] \tag{43}$$

$$\frac{\partial^2 p}{\partial x \partial y} = \mu \left[-\frac{\partial^4 \psi}{\partial x^2 \partial y^2} - \frac{\partial^4 \psi}{\partial x^4} - \frac{\partial^4 \psi}{\partial z^2 \partial x^2} \right] \tag{44}$$

Subtracting them we obtain

$$2\mu \frac{\partial^4 \psi}{\partial x^2 \partial y^2} = 0$$

implying that

$$\nabla^4 \psi = 0 \qquad (45)$$

Equation (45) shows that the stream function of a plane creeping flow is thus a *bi – potential* or *bi- harmonic* function. Some of the areas where small Reynolds number flows are considered are in the flow through a porous media , the Helle - Shaw flow and the lubrication theory. This chapter will deal more on the flows through porous media.

2.2.2 Flow Through a Porous Media

This kind of flow is one class of small Reynolds number flows. They are very prevalent in nature, and therefore requires a thorough investigation.

A porous media is literally a solid which contains a number of small holes distributed throughout the solid. The holes may be effective or ineffective. Effective holes are holes through which the fluid can actually pass, they contribute towards the porosity of the material. By ineffective holes it means those holes through which the fluid cannot pass through due to surface tension. Holes are usually taken to be or to mean effective. These holes may be distributed homogeneously or having any set pattern or may be very small or moderately large.

Examples of porous media include :cotton and woollen packings, wood dust, a pack of sand ,soil , sandstone , sponches, foamed plastics etc. The porosity of a material is defined as the fraction of the total volume of material occupied by the effective holes.

Porosity (s) is actually defined as:

$$s = 1 - \frac{\rho_d}{\rho_s}$$

where ρ_d is the density of dry porous material and ρ_s is the density of pore – free material.

The study of flows through porous media is comparably easy because in these flows, the inertia forces are very small compared to the viscous forces. Filtration of fluids , seepage of water in river beds , movement of underground water and oils , and seepage under water dams are examples of flows through porous media.

For instance an oil reservoir mostly consist of sedimentary formation such as limestone and sandstone in which oil is entrapped. Oil can be obtained from such reservoirs by drilling wells in the oil- bearing area down to the oil reservoir and then either allowing or causing the oil to flow through the porous oil – bearing rocks into the well, after which the oil can be pumped out of the well.

Also the study of flow through porous media is important in the study of underground water resources, in the sense that wells can be drilled and the water flows from the water bearing rocks into the well, and later pumped out from the well. The study of underground water resources may be applied in checking floods since the land along the sides of the river is usually porous and water goes underground due to seepage. In rainy season, the flow of water in rivers especially seasonal rivers is high, while in other seasons it is pretty low, one can therefore device ways of pumping water in the off season for irrigation purposes. This will lower down the level of underground

water in the riverbed area and during the rainy season, the river water will go underground with a greater seepage velocity and thus can check the floods.

Seepage under a dam with high pressure on the water side of the dam is also another field in which it involves flow through a porous media. The water seepage through the soil in the dam area lowers the water head and thus results in loss of energy. The study of seepage of water under the dam is thus very important.

2.2.3 Darcy's Law

From Darcy's experiment (1856), the rate of percolation of water is directly proportional to cross-sectional area of the filter bed and the total force ; say the sum of the pressure gradient and the gravitational force. The empirical formula obtained known as Darcy's law is

$$q = CA\left(\frac{p_1 - p_2}{h_1 - h_2} + \rho g\right) \tag{46}$$

where A is the cross-sectional area of the filter bed , $C = \frac{k}{\mu}$ in which k is the permeability , μ is the coefficient of viscosity and q is the flux of the fluid.

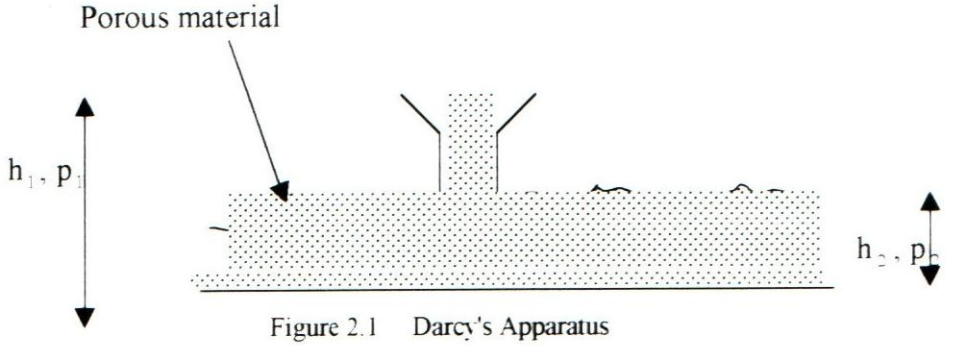


Figure 2.1 Darcy's Apparatus

Since this law is empirical, we have to generalize the above equation (46) to have a relation for a variable thickness of the porous material . A more generalized equation is

$$\vec{v} = -\frac{k}{\mu}\left(\nabla p + \rho g \vec{n} \right) \tag{47}$$

where $\vec{v}$ is the velocity vector and $\vec{n}$ is the unit vector along gravitational force taken along negative Z – axis.

Further if there exists an external force $\vec{F}$, instead of the gravitational force, then we can write

$$\vec{v} = \frac{-k}{\mu}\left(\nabla p - \rho \vec{F}\right) \tag{48}$$

since the pressure gradient ,the flux and the coefficient of viscosity can be determined experimentally, then one can use the flow through a horizontal slab for measuring the permeability of the porous material. Since in nature ,the gravitational force is an important force, we shall then consider $\vec{F}$ to be the gravitational force only.

The total potential ϕ of the pressure gradient and the gravitational force is defined as

$$\phi = \int \rho^{-1}\left[\nabla p + \nabla \Omega\right] ds$$

$$= \Omega + \int \rho^{-1} dp$$

$$= gz + \frac{p}{\rho} \tag{49}$$

Where $\Omega\left(\vec{F} = -\nabla\Omega\right)$ is the gravitational potential gz

Thus we define the velocity vector $\vec{v}$ in terms of the total potential as

$$\vec{v} = -\frac{\rho k}{\mu}\nabla\phi \tag{50}$$

here k, the characteristic of porous material is taken as a scalar essentially due to isotropicity of porosity.

Using the mass conservation equation, that is

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho(\vec{v})$$

and substituting into equation (50), we obtain

$$\frac{\partial \rho}{\partial t} = -\nabla\left(-\rho^2 \frac{k}{\mu}\nabla\phi\right)$$

$$= \nabla K \nabla\phi \tag{51}$$

where $K = \dfrac{\rho^2 k}{\mu}$

taking the flow to be steady, then equation (51) reduces to

$$\nabla \cdot (K\nabla\phi) = 0 \tag{52}$$

taking K as a constant (52) reduces to

$$\nabla^2\phi = 0 \tag{53}$$

which is the Laplace equation and can be solved by standard methods for a variety of boundary conditions. In two-dimensional flows through porous material, we can use the complex variable technique. We can introduce the complex functions ϕ and ψ, and the complex potential W = $\phi + i\psi$ which will be analytic function and the

theory of flows with calculation can be applied. In practice, we use electrical devices to investigate the complex problems of flow through porous media, which is mathematical modeling.

The above analysis of flow through porous material is simple despite the following limitations:

(i) it assumes that there is no interaction between the fluid and the porous material

(ii) the fluid velocity is taken to be small so that inertial force can be neglected.

(iii) The fluid is taken to mean a liquid. Gases give a marked departure from the results.

Examples of flows through porous media include: - flow through a porous slab, flows through a porous annulus, seepage under a dam and flows through a spherical shell. This chapter will put emphasis on discussion of these flows with an aim of giving some of the applications of small Reynolds number effects on fluid flows.

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2.2.4 Flow Through a Porous Slab

Consider an infinite horizontal slab of thickness d , confined between the planes z = 0 and z = d as shown by figure 2.2 below

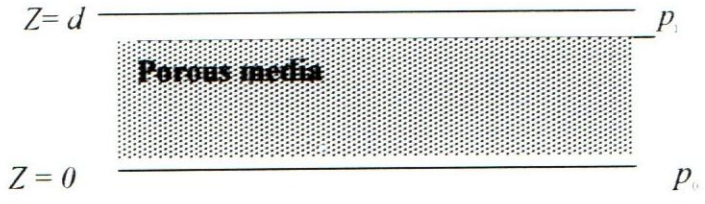


Figure 2.2 an infinite porous horizontal slab

If pressure at $z = 0$ is P_o and that at $z = d$ is P_1 , taking the flow to be vertical then

$$\frac{d^2\phi}{dz^2}=0 \tag{54}$$

On integration we obtain

$$\phi = AZ - B \tag{55}$$

where A and B are constants.

But potential ϕ is also given by

$$\phi = gz + \frac{p}{\rho} \tag{56}$$

hence $AZ - B = gz + \frac{p}{\rho}$

$$\Rightarrow p = \rho(AZ + B - gZ) \tag{57}$$

using the boundary conditions on equation (57) we obtain

$$A = g + \left(\frac{p_1 - p_0}{\rho d} \right) \tag{58}$$

and $$B = \frac{p_0}{\rho} \tag{59}$$

thus the potential

$$\phi = \left[g + \left(\frac{p_1 - p_0}{\rho d} \right) \right] z + \frac{p_0}{\rho} \tag{60}$$

Substituting the value of ϕ from equation (60) into equation (50) and using the definition of ∇ leaving x and y components, we obtain the vertical velocity as

$$u = -\frac{k}{\mu} \left[\rho g + \frac{p_1 - p_0}{d} \right] \tag{61}$$

which is similar to Darcy's law.

Now, if the permeability is not constant but a function of Z, then we have from the equation $\nabla \cdot (K \nabla \phi) = 0$

or $$\frac{d}{dz} \left(K \frac{d\phi}{dz} \right) = 0 \tag{62}$$

On integration yields

$$\phi = A \int \frac{dz}{k} + B \tag{63}$$

Where A and B are constants of integration

but ϕ is also given by

$$\phi = \frac{p}{\rho} + gz$$

$$\Rightarrow p = \rho(\phi - gz) \tag{64}$$

Substituting equation (63) into (64) we obtain

$$p = \rho\left(A\int\frac{dz}{k} + B - gz \right)$$

Using the boundary conditions i.e. at $Z = 0$, $P = P_0$, and at $Z = d$, $P = P_1$ we obtain

$$p_0 = \rho \cdot B \qquad \Rightarrow \qquad B = \frac{p_0}{\rho}$$

and

$$A = \left(\frac{p_1 - p_0}{\rho d\int\frac{1}{k}} \right) + gd^2\int\frac{1}{k}$$

hence the potential Φ becomes

$$\phi = \left[\frac{p_1 - p_0}{\rho d\int\frac{1}{k}} + gd^2\int\frac{1}{k} \right]\int\frac{dz}{k} + \frac{p_0}{\rho} \tag{65}$$

2.2.5 Flow Through a Porous Annulus

Considering the seepage of a fluid through a horizontal porous annulus of radii a and b ($b > a$), and using cylindrical coordinate system , let the pressure at $r = a$ be P_0 and at $r = b$ be P_1 . Taking the flow to be steady, permeability to be constant and the flow dependent on r, we see that equation (53) becomes

$$\frac{d}{dr}\left[r\frac{d\phi}{dr}\right]=0 \tag{66}$$

Which on integration yields

$$\phi = A + B \log r \tag{67}$$

but also $\phi=\frac{p}{\rho}+gz$, but $gz = 0$ since the flow is radial. Hence the potential becomes

$$\phi=\frac{p}{\rho} \tag{68}$$

thus at $r = b$

$$\phi=\frac{p_1}{\rho}$$

and at $r = a$, $\phi=\frac{p_0}{\rho}$

therefore

$$\frac{p_1}{\rho}=A+B\log b$$

$$\frac{p_0}{\rho}=A+B\log a$$

$$\Rightarrow B=\frac{p_1-p_0}{\rho}\frac{1}{\log\left(\frac{b}{a}\right)} \tag{69}$$

and

$$A=\frac{p_0}{\rho}-B\log a \tag{70}$$

hence the potential ϕ becomes

$$\phi = \frac{1}{\rho}\left[p_0 + (p_1 - p_0)\log\left(\frac{r}{a}\right) \cdot \left(\frac{1}{\log(\frac{b}{a})} \right) \right] \tag{71}$$

Substituting equation (71) into the velocity vector equation $\vec{v} = -\frac{\rho k}{\mu}\nabla\phi$, we obtain

$$\vec{v} = \frac{K(p_1 - p_0)}{\mu r \log\left(\frac{b}{a}\right)} a \tag{72}$$

and the flux (q) per unit length of the annulus is defined as

$$q = v.2\pi r$$

$$= 2\pi k(p_1 - p_0) \cdot \left(\frac{a}{\log\left(\frac{b}{a}\right)} \right) \tag{73}$$

2.2.6 Flow Through a Spherical Shell

Considering the flow of a fluid through a porous spherical shell with inner and outer radii of a and b respectively and taking the permeability to be proportional to the radial distance (i.e. $k = k_o r$) and the pressure at $r = a$ be p_o and at $r = b$ be P_1 (say atmospheric pressure), then we have the equation $\nabla^2\phi = 0$ becoming

$$\frac{d}{dr}\left(kr^2 \frac{d\phi}{dr} \right) = 0 \tag{74}$$

since the flow is purely radial.

But $k = k_0 r$, hence equation (74) becomes

$$\frac{d}{dr}\left(k_0 r^3 \frac{d\phi}{dr}\right) = 0$$

Which on integration yields

$$\phi = \frac{A}{r^2} + B \tag{75}$$

where A and B are constants of integration

Substituting this value of ϕ in equation $\phi = gz + \frac{p}{\rho}$ and neglecting gravitational force since the flow is totally radial and using the boundary conditions on the pressure P we obtain

$$p = p_0 + \frac{(p_1 - p_0)a^2 b^2}{b^2 - a^2}\left(\frac{1}{a^2} - \frac{1}{r^2}\right) \tag{76}$$

Substituting this value of P from equation (76) into the velocity equation $\vec{v} = \frac{-\rho k}{\mu}\nabla\phi$ with $\phi = \frac{p}{\rho}$ we get the velocity towards the center of sphere as

$$\vec{v} = \frac{2k_0 a^2 b^2 (p_1 - p_0)}{\mu r^2 (b^2 - a^2)} \tag{77}$$

and the total flux (q) in this case is given by

$$q = \vec{v}.4\pi r^2$$

$$= \frac{8\pi k_0 a^2 b^2 (p_1 - p_0)}{\mu(b^2 - a^2)} \tag{78}$$

2.2.7 Seepage Under a Dam

A more realistic situation is the seepage under an earth dam. The soil is taken as a porous material and thus the water will flow through the soil. Consider the figure 2.3 below

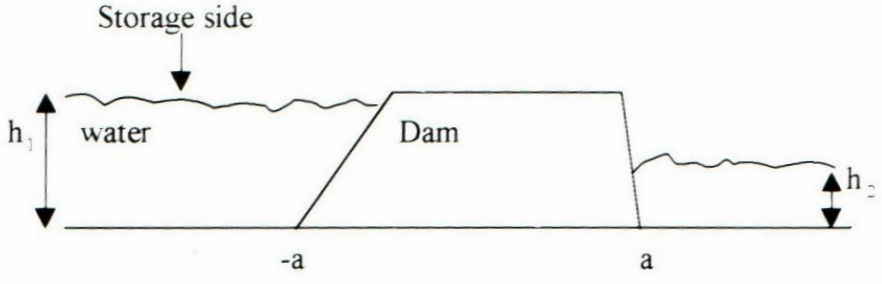


Figure 2.3 Structure of a dam

The soil is taken for simplicity as a semi – infinite region.

If the depth of the water at storage side is taken as h_1 and on the other side is h_2 as shown above, and the dam is made up of concrete ,then we can assume that water cannot percolate below it (soil). Thus the porous media (soil) is semi – infinite with a sealed strip between $x = -a$ and $x = a$.

Now the potential ϕ at $x < -a$ is given by

$$\phi = gh_1 \tag{79}$$

and at x > a is given by

$$\phi = gh_2 \tag{80}$$

Using the complex variable technique , the porous region is $y < 0$, thus we can take

$$\varsigma = \xi + i\eta = \frac{1}{g(h_1 - h_2)}\left[\phi + i\psi - gh_2\right] \tag{81}$$

and to transform $z(=x+iy)$ plane to the $w(=u+iv)$ plane by the transformation

$$w = \sin^{-1}(z/a). \tag{82}$$

The porous region is thus transformed into a column $\frac{-\pi}{2} < u < \frac{\pi}{2}$, $v < 0$ and the governing equation become

$$\frac{\partial^2 \xi}{\partial u^2} + \frac{\partial^2 \xi}{\partial v^2} = 0, \qquad \frac{\partial^2 \eta}{\partial u^2} + \frac{\partial^2 \eta}{\partial v^2} = 0 \tag{83}$$

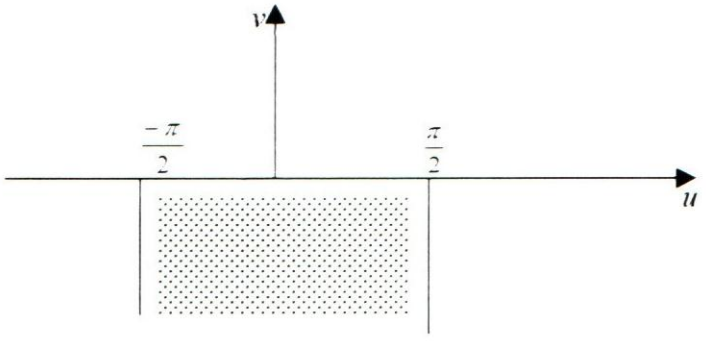


Figure 2.4 Porous region transformed into w- plane

The boundary conditions are

$$\xi = 1 \text{ for } u = \frac{\pi}{2},\ v < 0 \text{ and } \xi = 0 \text{ for } u = \frac{-\pi}{2},\ v < 0 \tag{84}$$

hence the solution of equation (83) becomes

$$\left.\begin{aligned} &\frac{\partial^2 \xi}{\partial u^2} = 0 \ ; \Rightarrow \xi = Cu + K \\ &\frac{\partial^2 \eta}{\partial u^2} = 0 ; \Rightarrow \eta = Cu + D \end{aligned}\right\} \tag{85}$$

and using boundary conditions (84) on equations (85) we obtain

$$\xi = \frac{1}{2} + \frac{u}{\pi} \quad , \quad \eta = \frac{v}{\pi} \tag{86}$$

returning to the Z- plane , the solution now becomes

$$\phi = \frac{1}{2} g(h_1 + h_2) + \frac{1}{\pi} g(h_1 - h_2) \sin^{-1} \left[\frac{1}{2a} \left\{ \sqrt{(a-x)^2 + y^2} \right\} - \sqrt{(a+x)^2 + y^2} \right] \tag{87}$$

The seepage velocity for $|x| > a$ is therefore given by

$$v_{y=0} = \frac{-\rho k}{\mu} \frac{d\phi}{dy}$$

Solving it we obtain

$$v_{y=0} = \frac{-\rho g k}{\pi \mu} (h_1 - h_2) \cdot \frac{1}{\sqrt{x^2 - a^2}} \quad , \quad x \neq 0 \tag{88}$$

The total flux (q) for $x = a$ to x due to seepage therefore becomes

$$q = \int_a^x v dx$$

$$q = \frac{\rho k g}{\pi \mu} (h_1 - h_2) \cosh^{-1} \left(\frac{x}{a} \right) \tag{89}$$

2.3 Conclusion

From these analyses or discussions on flows through a porous media,i.e seepage under a dam, flow through a porous annulus , and flow through a spherical shell, we can see that Reynolds Number affects the flow characteristics of the particular kind of flow such as the velocity(v) and the total flux(q) of the fluid in consideration. The above examples demonstrate the flows in which their Reynolds Number is small, that's the creeping flows.

CHAPTER THREE

LARGE REYNOLDS NUMBER FLOWS

The Boundary Layer Flows

3.1 Introduction

A complete solution of the Navier – Stokes equations has not been accomplished to date because when friction and inertia forces are of the same order of magnitude in the entire flow system, then neither of them can be neglected. Some exact solutions have been found by making the Navier-Stokes equations linear by taking a simple geometry of flow and assuming that the fluid is incompressible. In chapter two, approximate solutions of Navier – Stokes equations for very small Reynolds number flows were dealt with. In this cases frictional forces over-shadowed the inertia forces and the equations become linear by omitting the convective acceleration. However, in this chapter we shall discuss the flow characterised by very large Reynolds numbers. This means that the non-linear inertia terms of the Navier – Stokes equations are far greater than the viscous terms. On this consideration it seems logical to ignore the viscous terms, but we cannot do this because the order of these differential equations (N – S) is reduced by neglecting the terms containing viscosity and the boundary condition of no –slip which requires that the velocity be equal to zero on all rigid surfaces bounded by the fluid is violated for the simplified equations of an ideal fluid obtained by ignoring the viscous terms. Physically the adhesion of the fluid to the solid surface takes place no matter how small the viscosity happens to be. Hence in viscous flow of large Reynolds number, the flow at a distance

from the rigid wall becomes similar to the ideal flow in which viscosity is considered to be zero, but for the flow close to the wall, a thin layer exists in which there is high shear. That is, the velocity of the flow changes very rapidly from the value of zero at the wall to the value existing on the outer boundary of the layer. This value is equal to the velocity obtained in the case of ideal fluid. This thin layer close to the wall with high shear is called the *boundary layer*, and in this layer the effect of frictional forces is of the same order as the effect of inertia forces despite the large Reynolds number.

The concept of boundary layer plays a vital role in fluid dynamics. It is a powerful method of analysing the complex behaviour of real fluids. It can be utilised to simplify the Navier – Stokes equations to such an extend that it becomes possible to tackle many practical problems of great importance such as drag on airfoils, efficiency of compressors and turbines in jet engines, the efficiency of air intakes for ram and turbojets, prediction of separation and explanation on the existence of wakes as shall be discussed in chapter five of this dissertation.

The success or failure of a craft is directly related to its resistance to motion. If the drag is too large, the craft may be an economic failure. The drag on a body results from the shear and pressure forces existing or acting on the body. The shear stress on a smooth plane is a direct function of velocity gradient next to the plate. Hence a function of the Reynolds number. Therefore, any problem involving the shear stress is also involved with the flow pattern in the vicinity of the surface. (I.e. boundary layer flow)

3.2 Laminar Boundary Layer on a Flat Plate

Consider a boundary layer on a flat plate of length L and infinite width lying on $0xy$ plane, so that it occupies the region $0 \leq X \leq L$ of the plane and flow past with velocity v_1 in the 0_x direction. Here the characteristic length and velocity scales are given by L and U respectively, so the no – linear terms of the Navier-Stokes equations is of order $U\,2L$ in the boundary layer. At the same time the longitudinal velocity v_1 varies in the boundary layer from $v_1 = 0$ at the surface of the plate to a value of the order U on the outer edge of the layer as shown by figure 3.1 below

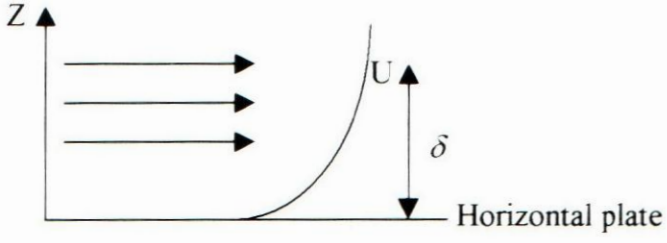


Fig 3.1 Velocity profile of a flat plate

Let the thickness of the boundary layer be δ. This thickness is not well defined at the outer edge, since the velocity v_1 tends asymptotically to U with increasing distance from the plate in the 0_z direction. Therefore we consider δ to denote the distance from the plate in 0_z direction at which the velocity v_1 attains a given, sufficiently great fraction of U.

Now, we are in a position to get $\frac{\partial^2 v_1}{\partial z^2}$ to be equal to the order of U/δ^2 and the viscous term $v\frac{\partial^2 v_1}{\partial z^2}$ is of the order of vu/δ^2. Already it has been established that the viscous forces are of same importance as the inertia forces in the boundary layer.

$$\frac{U^2}{L} \sim \frac{\nu U}{\delta^2}$$

$$\text{or} \quad \delta \sim \left(\nu L/U\right)^{\frac{1}{2}} = L/\text{Re}^{\frac{1}{2}} \; ; \; \text{Re} = \frac{UL}{\nu} \tag{90}$$

Equation (90) shows that the thickness of the boundary depends on the Reynolds number (*Re*) of the flow and it is thinner for large Reynolds numbers that are obtained by the smaller viscosity of the fluid and the higher free stream velocity. If the thickness of the boundary layer at a distance X from the leading edge of the plate is required, the total length L is replaced by the length X, then it results in

$$\delta \sim \left(\frac{\nu x}{U}\right)^{\frac{1}{2}} \text{ at the point } X, \tag{91}$$

This means that the thickness of the boundary layer increases in proportion to $x^{\frac{1}{2}}$. The shear stress at the wall is given by $\tau = \mu \frac{\partial v_1}{\partial z}\Big|_{z=0}$. The gradient $\frac{\partial v_1}{\partial z} \sim \frac{U}{\delta}$, then $\tau \sim \mu \frac{U}{\delta}$ or in view of equation (90)

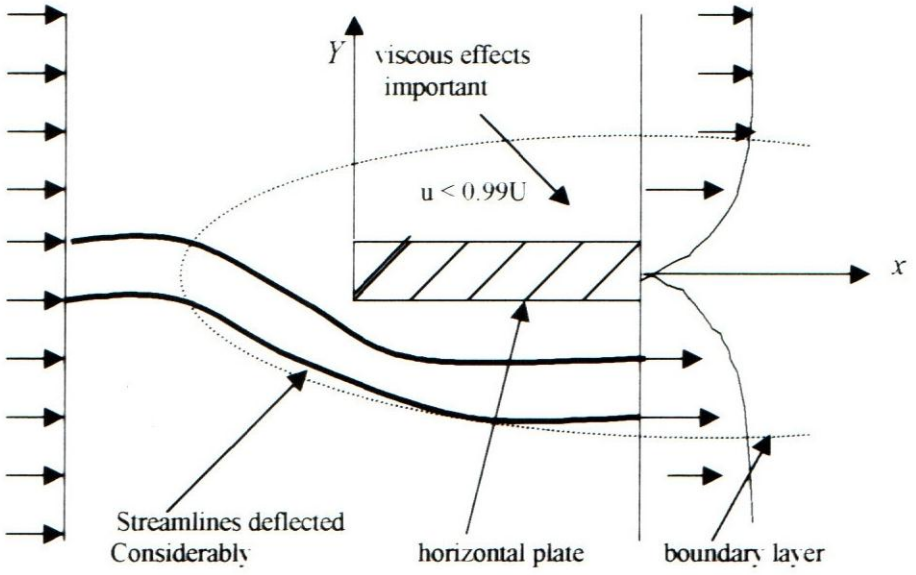
$$\tau \sim \rho \left(\nu \frac{U^3}{L}\right)^{\frac{1}{2}} \tag{92}$$

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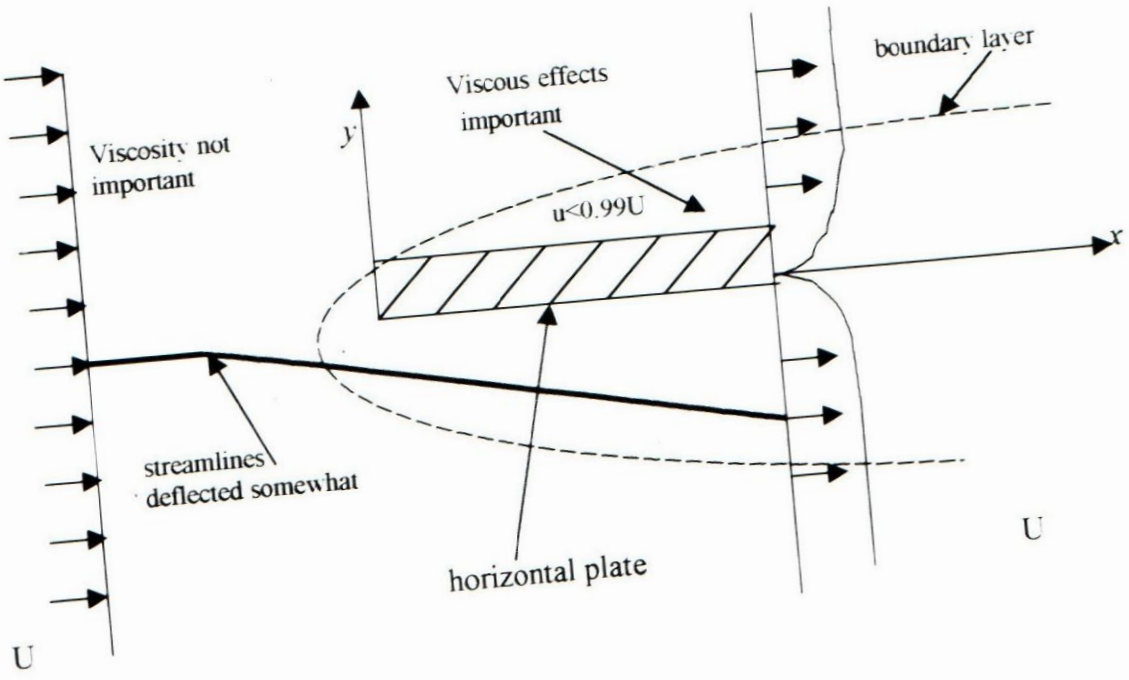
Here the shear stress is proportional to $\frac{U^3}{2}$ and the corresponding skin friction coefficient is given by $C_f = \frac{\tau}{\frac{1}{2}\rho U^2} \sim \frac{1}{(\mathrm{Re})^{\frac{1}{2}}}$ (93)

The skin friction coefficient C_f, of course depends on the Reynolds number (Re) and decreases as Re increases as illustrated by equation (93)

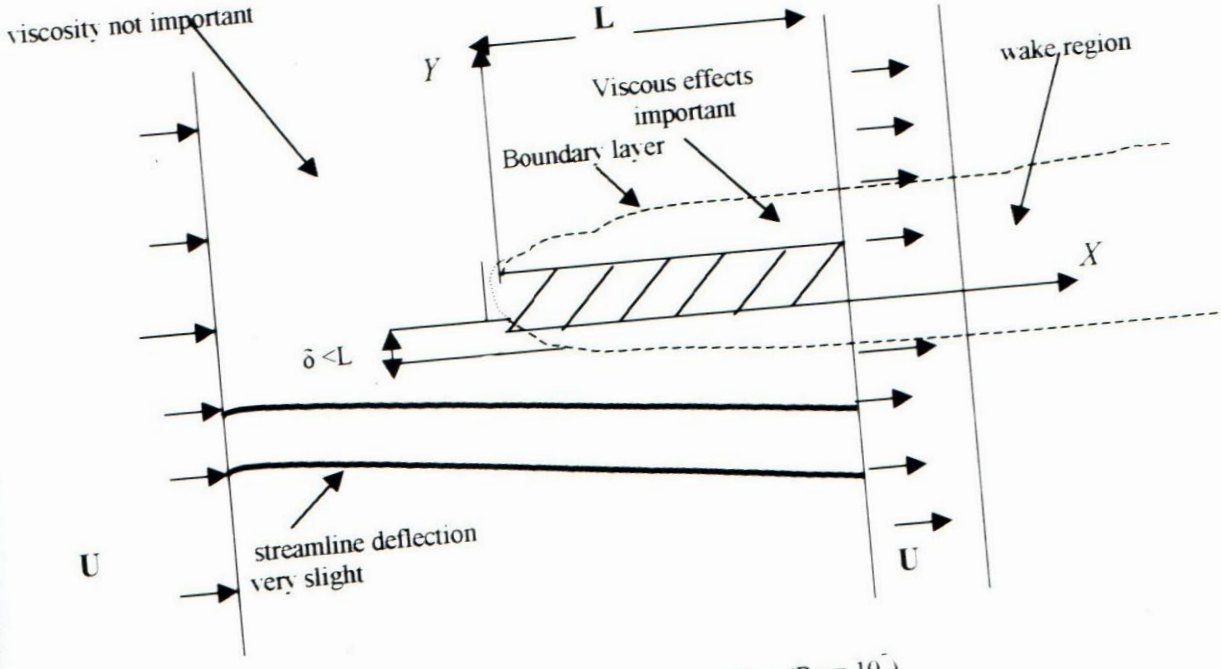
The Reynolds number affects the flow over a flat plate parallel to the upper stream velocity U, in the sense that the boundary layer thickness (δ), the skin friction coefficient C_f and the streamlines of the flow are affected by changes in the value of the Reynolds number as illustrated figures 3.2 (a) , (b) and (c) below



3.2 (a) small Reynolds number flow (Re ~ 0.1)



3.2 (b) moderate Reynolds number flow ($Re = 10$)



3.2 (c) Large Reynolds number Flow ($Re = 10^{-}$)

Figures. 3.2 : Character of steady. viscous flow past a flat plate to upstream velocity

The characteristics described by figures 3.2 for flow past a flat plate are typical of flows past a streamlined body. The nature of flow depends strongly on the Reynolds number. Most familiar flows are similar to the large Reynolds number flows depicted by figure 3.2c rather than the low Reynolds number flow situations.

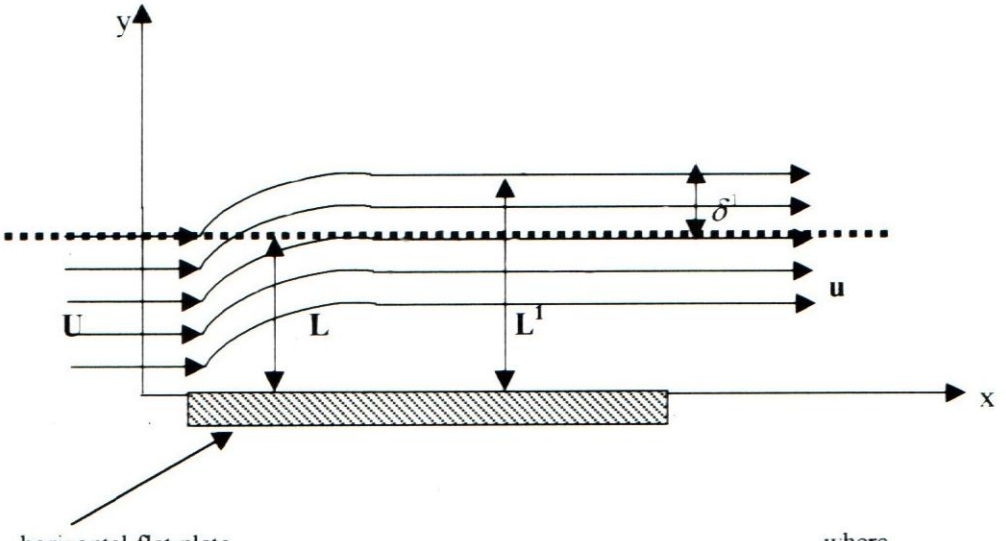
3.3 Displacement, Momentum and Energy Thickness

Sometimes we require only the rough estimate of the boundary layer flow over any object. Mainly we may be concerned about the skin friction, drag, the momentum transfer and energy transfer in the boundary layer region and not in the velocity distribution region. Moreover, the boundary layer thickness is very sensitive to the decision we take about the distance L from the body as illustrated by equation (90), so we consider other types of thickness which are not sensitive to this length. These are the displacement, momentum and energy thickness.

3.3.1 Displacement Thickness (δ_1)

This is the distance the streamlines are displaced due to the presence of the body. Considering a uniform flow over a flat plate, at a far distance from the plate the flow should be unaffected due to the presence of the plate and therefore the velocity component parallel to the plate should be the same as the upstream flow velocity. Moreover due to retardation in the boundary layer region, the velocity near the plate is less than the upstream flow velocity, say U and so the flux over the plate through a distance d will be less than the flux through the same cross – section in the upstream region. Thus the principle of conservation of mass gives that to get the same flux over the

plate as in the upstream region through the distance d, we have to take, while considering the flow on the plate, a little more distance than d which shows as if the streamlines have been displaced by small distance δ_1. This distance is the *displacement thickness*. To get an estimate of displacement thickness, we take a stream at a far distance from the plate along y – axis and calculate the flux between this streamline and the streamline $y = 0$.



where,
U- Main flow velocity
u- Velocity near the plate surface
δ_1 – displacement thickness

Figure 3.3 illustration of momentum thickness

Thus we have $\int_o^L U dy = \int_o^L u dy + \int_L^{L^1} u dy$ (94)

Where L and L^1 are respectively the distance chosen from streamline $y = 0$ in the upstream region and over the plate as shown by fig. 3.3.

From equation (94), we have

$$U(L^1 - L) = \int_o^L (U - u) dy$$

or $\delta_1 = \lim_{L^1 \to \infty} (L^1 - L) = \lim_{L^1 \to \infty} \int_o^L \left(1 - \frac{u}{U}\right) dy$

or $\delta_1 = \int_o^y \left(1 - \frac{u}{U}\right) dy$. (95)

hence displacement thickness represents the amount that the thickness of the body must be increased so that the fictitious uniform inviscid flow has the same mass flow rate properties as the actual viscous flow.

3.3.2 Momentum Thickness (δ_2)

In flow over a fat plate, the streamlines are displaced away from the plate due to retardation in the boundary layer region, as a result there is a loss of momentum in the boundary layer region. The momentum transfer along the plate through a particular cross-section is less than in the upstream region. This loss of momentum is transferred to the plate. The distance through which the momentum is pushed away from the plate is called the *momentum thickness*.

If δ_2 is the momentum thickness, then the momentum passing through this cross – section is equal to the loss of momentum in the boundary region given by;

$$\rho(U - u) \tag{96}$$

where $(U - u)$ is the velocity change

hence, the rate of loss of momentum becomes $\rho(U - u)u$ and thus the total rate of change of loss of momentum becomes

$$\int_o^\delta \rho u(U - u) dy \tag{97}$$

where δ is the boundary layer thickness. Since there is no retardation in the flow outside the boundary layer, then δ can be replaced by ∞. Thus this must be equal to the momentum flux through the momentum thickness. That is

$$\int_0^{\delta} \rho u(U-u)dy = \rho u^2 \delta_2$$

where $\rho u^2 \delta_2$ is the momentum flux.

Hence $\delta_2 = \int_0^{\delta} \frac{u}{U}\left(1-\frac{u}{U}\right)dy$ (98)

Momentum thickness is useful in evaluating the drag along a distance x of a plate as well as the shear stress at the surface of the plate or any object in motion.

3.3.3 Energy Thickness (δ_3)

There is also energy loss in the boundary layer region due to retardation of the fluid motion. Energy thickness (δ_3) is defined as the thickness of a layer with upstream velocity U which has the same kinetic energy flux as the rate at which kinetic energy is being lost in the boundary layer region. The loss in kinetic energy at a distance y from the plate is given by

$$\frac{1}{2}\rho\left(U^2-u^2\right)$$

Rate of loss of kinetic energy becomes

$$\frac{1}{2}\rho u\left(U^2-u^2\right)$$

Then the total rate of loss of kinetic energy is therefore

$$\frac{1}{2}\int_0^{\delta} \rho u\left(U^2-u^2\right)dy \qquad (99)$$

The kinetic energy flux is then given by

$$\frac{1}{2}\rho u^3 \delta_3 \qquad (100)$$

hence the kinetic energy loss equals the kinetic energy flux, i.e.

$$\frac{1}{2}\int_0^{\infty} \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy = \frac{1}{2}\rho U^3 \delta_3 \qquad (101)$$

thus the energy thickness from equation (101) becomes

$$\delta_3 = \int_0^{\infty} \frac{u}{U}\left(1-\frac{u^2}{U^2}\right)dy \qquad (102)$$

All these boundary layer thickness vary with the magnitude of the Reynolds number of the flow. Large Reynolds number, means a decrease in boundary layer thickness and for small Reynolds number these thickness increase in size. These thickness play a very important role in determining motion of airfoils and flow over immersed bodies because it affects the drag , the lift and the skin- friction on this objects.

3.4 Skin Friction and Drag in a Laminar Boundary Layer Flow

The local skin – friction coefficient C_f is defined as

$$c_f = \frac{shear stress}{unit\ kinetic\ energy} = \frac{\tau_w}{\frac{1}{2}\rho u^2}$$

c_f is dimensionless and is a kind of Euler number . Using the approximate profile used in the Karman momentum integral approximation, we have for c_f using Newton's viscosity law as

$$c_f = \frac{\mu\left(\frac{\partial u}{\partial y}\right)y=o}{\frac{1}{2}\rho U^2} = \frac{\mu\left\{\frac{\partial}{\partial y}\left[2U\frac{y}{\delta} - U\left(\frac{y}{\delta}\right)^2\right]\right\}_{y=o}}{\frac{1}{2}\rho U^2}$$

$$c_f = \frac{\mu 2U\left(\frac{1}{\delta}\right)}{\frac{1}{2}\rho U^2} \qquad (103)$$

but $\delta = \frac{L}{\sqrt{\mathrm{Re}}}$ from Equation (90)

hence $c_f = \frac{\mu 2U\left(\frac{\sqrt{\mathrm{Re}}}{L}\right)}{\frac{1}{2}\rho U^2}$

$$= \frac{4\mu\sqrt{\mathrm{Re}}}{L\rho U}$$

$$= \frac{4\nu\sqrt{\mathrm{Re}}}{UL} = \frac{4\sqrt{\mathrm{Re}}}{\mathrm{Re}}$$

$$c_f = 4\mathrm{Re}^{-\frac{1}{2}} \tag{104}$$

This indicates that Reynolds number (*Re*) is a function of local skin-friction. The greater the Reynolds number, the lower the local skin-friction and vice versa.

The drag on one side of the whole plate of say width b and length ℓ having a laminar boundary layer is then given as

$$D = \int_A \tau_w dA = \int_o^l c_f \left(\frac{1}{2}\rho U^2\right) b dx \tag{105}$$

but from Blasius solution , $c_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$; $\mathrm{Re}_x = \frac{Ux}{\nu}$

substituting this into equation (105) gives

$$D = \int_o^L \frac{0.664}{\sqrt{\frac{Ux}{\nu}}}\left(\frac{1}{2}\rho U^2\right) b dx$$

$$= \frac{0.664}{\sqrt{\frac{U}{\nu}}}\left(\frac{1}{2}\rho U^2\right) b \int_o^l x^{-\frac{1}{2}} dx$$

$$D = 0.664 b\sqrt{LU^3\mu\rho} \tag{106}$$

This equation (106) shows that the drag depends on length of plate $l^{\frac{1}{2}}$. This means that the drag towards the end of the plate contributes less proportionately than the drag near the leading edge of the plate. This is due to the fact that the boundary layer is thicker at the end, giving a smaller slope of the velocity profile at the plate boundary. This in turn results in a smaller shear stress.

The coefficient of skin friction for the whole plate called the plate skin friction coefficient denoted as C_f is defined as

$$C_f = \frac{D}{\frac{1}{2}\rho U^2 A} \tag{107}$$

where D is Drag force

Using equation (106) in (107) we get the plate skin friction coefficient as

$$C_f = \frac{(0.664)b\sqrt{lu^3\mu l}}{\left(\frac{1}{2}\rho U^2\right)(bl)} = 1.328\left[\frac{\nu}{Ul}\right]^{\frac{1}{2}}$$

hence we get $C_f = 1.328/\sqrt{\mathrm{Re}L}$ (108)

where Re is the plate Reynolds number. This equation shows that, the plate skin friction coefficient is a function of Reynolds number, hence its significance.

3.5 Transition from Laminar to Turbulent flow in the Boundary Layer for a Flat Plate

The parameter that governs the transition from laminar to turbulent flow or turbulent to laminar flow is the Reynolds number, i.e. the Reynolds number based on the distance from the leading edge of the plate,

$$Re_x = \frac{Ux}{\nu}$$

The value of Reynolds number at the transition location is rather a complex function of many parameters involved, including the roughness of the surface, the curvature of the surface (e.g. a flat plate or a sphere), and some measure of the disturbances in the flow outside the boundary layer.

The actual transition from laminar to turbulent boundary layer flow may occur over a region of the plate, not at a specific single location. This occurs, in part, due to the spottiness of the transition. The transition begins at random locations on the plate in the vicinity of $Re_x = Re_{x_{cr}}$, where $Re_{x_{cr}}$ is the critical Reynolds number based on the distance x from the leading edge of the flat plate. These spots grow rapidly as they are convected downstream until the entire width of the plate is covered with turbulent flow.

The complex process of transition from laminar to turbulent involves the instability of the flow field. Small disturbances imposed on the boundary layer (i.e. from plate vibration, roughness of the surface, or a 'wiggle' in the flow past the plate) will either grow (instability) or decay (stability), depending on where the disturbances is introduced in the flow. If the disturbances occur at a location with $Re_x < Re_{x_{cr}}$ they will die out, and the boundary layer will return to laminar flow at that location. Disturbances imposed at a location with $Re_x > Re_{x_{cr}}$ will grow and transform the boundary layer flow downstream of this location into turbulence.

Transition from laminar to turbulent flow also involves a noticeable change in the shape of the boundary layer velocity profile. The turbulent profiles are flatter, have a larger

velocity gradient at the wall, and produce a larger boundary layer thickness than do the laminar profiles as shown by figure 3.4 below.

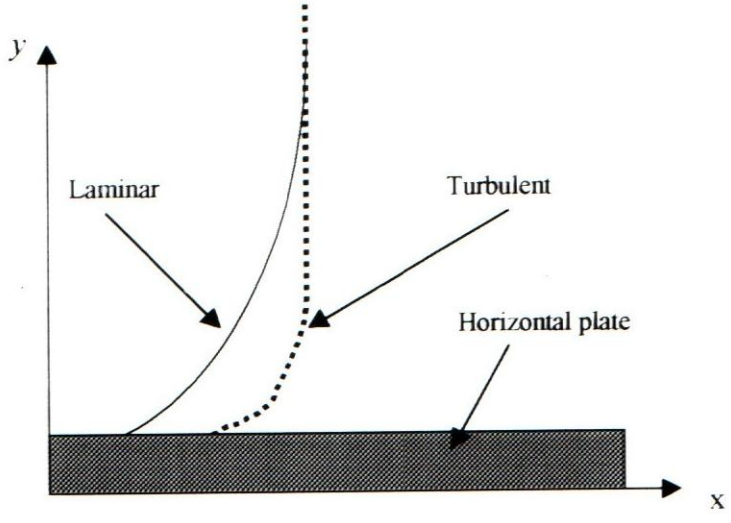


Figure 3.4 comparison of laminar and turbulent Velocity profiles in a boundary layer.

From this analysis, Reynolds number is of great significance in determining the transition from laminar to turbulent boundary layer flow.

The boundary layer thickness δ is known to be a function of Reynolds number, i.e.

$$\delta = f(\mathrm{Re}) \tag{109}$$

from equation(90)

or $\delta = f\left(\sqrt{\frac{\vartheta x}{U}}\right)$, since $\mathrm{Re}^{\frac{1}{2}} = \sqrt{\frac{\vartheta x}{U}}$

but the boundary layer remains laminar up to a distance x_{cr} from the edge of the plate, when it starts to change to turbulent. I.e.

$$x_{cr} = \frac{\vartheta \mathrm{Re}_{xcr}}{U} \tag{110}$$

where U is the upstream velocity of fluid

ϑ is the viscosity

and Re_{xcr} is the critical Reynolds number

Equation (109) illustrates the fact that we can be able to determine the distance the boundary layer will be turbulent and how thick the boundary layer will be approximated. Thus we can say that laminar flow can be maintained on a longer portion of a plate if the viscosity is increased. However, the flow eventually becomes turbulent as long as the plate is long enough. Similarly boundary layer thickness is greater if the viscosity is increased.

In summary, the Reynolds number (*Re*) determines the flow in the boundary layer region. This concept of boundary layer flow is significant in analysing motion airfoils and immersed bodies as shall be discussed in chapter five. The boundary layer flow theory is of great significance and hence the Reynolds number.

CHAPTER FOUR

LAMINAR AND TURBULENT FLOWS

4.1 Introduction

The influence of viscosity on flow is frequently very complex and not amenable to rigorous mathematical analysis. The shear stresses resulting from viscosity may be included in the equations of motion for a small element of fluid, but the resulting partial differential equations (Navier-Stokes equations) have no known general solution. For some years however, it had been realised that the flow of fluids could be of two quite different kinds. The work of Hagen in about 1840 served to emphasise certain consequences of these differences, however, it was until the experiments of Reynolds in the early 1880's that the essential nature of the two types of flow was clearly demonstrated.

The first kind of flow observed by Reynolds is that occurring at low velocities, the particles of the fluid are evidently moving entirely in straight lines even though the velocity with which particles move along one line is not necessarily the same as that along the other line. The fluid is considered to move in layers, or laminae. This kind of flow is usually called *Laminar flow*.

The second type of flow is known as *Turbulent flow*, where the paths of individual particles of the fluid are no longer everywhere straight, but are sinuous, intertwining and crossing each other in a disorderly manner, so that thorough mixing of the fluid takes place. Turbulent flow is essentially irregular motion on a small scale. It's a flow that is

complex, where a rigid mathematical analysis is impossible because of the erratic motion of the particles.

Therefore, whether a flow is laminar or turbulent depends on the properties of the particular flow.

4.2 Laminar and Turbulent Flow in Pipes

In the case of flow through a pipe or a closed conduit, the flow of the fluid is subject to inertia and viscous forces, apart from forces due to differences in piezometric pressure between the different parts of the fluid. The nature of the flow is normally determined by the value of the dimensionless parameter; the Reynolds number, $Re = \frac{\rho v D}{\mu}$, where ρ is the density of the fluid, v is the average flow velocity, D is the pipe diameter and μ is the viscosity of the fluid.

For circular pipes it is usual to take the diameter D as the linear measurement representative of flow pattern. (Note that for a pipe of uniform diameter, the length of the pipe is not relevant in determining the flow pattern.)

Laminar flow occurs at low velocities and therefore low values of Reynolds number, whereas turbulent flow takes place at high Reynolds number. This follows that for laminar flow, the viscous forces are predominant, but for turbulent flow, it is inertia forces that take the lead.

Laminar flow is unsteady flow regime, in that when the velocity is increased, eddies begin suddenly rather than gradually; so slight disturbance is sufficient to bring on fully turbulent flow. Critical velocity at which turbulence begins is very sensitive to any initial

conditions. The Reynolds number Re = $\frac{\rho v D}{\mu}$ is a chief factor determining the onset of turbulence. Under normal engineering conditions, where disturbances are always present, transition from laminar to turbulent flows occurs at approximate Reynolds number values of between 2000 and 4000. There is apparently no upper limit to the value of Reynolds number at which change occurs. However, there is a definite lower limit. When Reynolds number is less than this lower limit, any disturbance in the flow is damped out by the viscous forces, which are predominant.

There are two critical velocities at which transition from laminar to turbulent takes place as illustrated by the figure 4.1 below. We have the higher or upper critical velocity (point B) where transition takes place from laminar to turbulent and the lower critical velocity (point A) where transition takes place from turbulent to laminar flow. The lower critical value is of much interest and importance because it is the point at which laminar flow becomes unstable. It is usual to refer to this lower critical value as simply ***Critical Reynolds Number*** (Re_{cr})

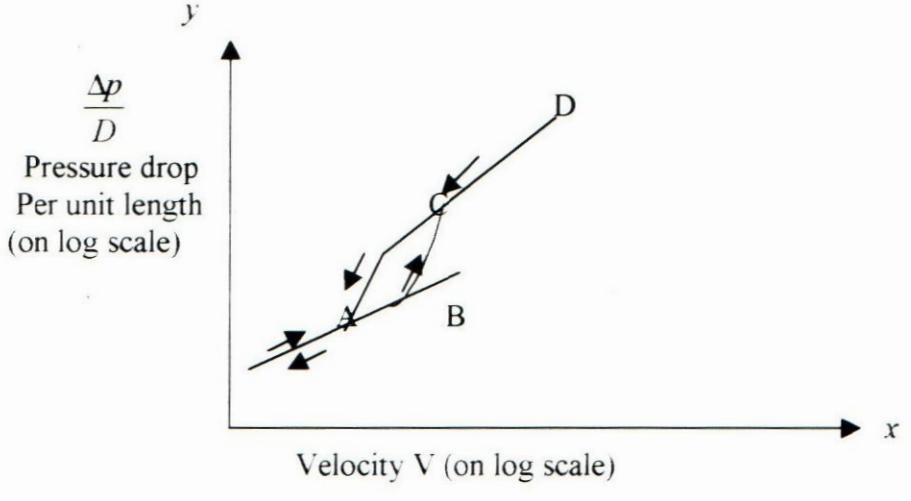


Figure 4.1 Line graph for two critical velocities at which transition from Laminar to Turbulent takes place

For ordinary purposes, it is usual to take critical Reynolds number as 2300, however, curvature of the pipe axis also reduces this figure. If the pipe converges, the critical Reynolds number is higher, whereas if it diverges, the value reduces.

If Reynolds number exceeds its critical value an originally turbulent flow will remain turbulent, but if the flow was originally laminar, it will tend to become turbulent if any disturbance is introduced. But if Reynolds number is less than its critical value, an originally laminar flow will remain laminar and if flow at the entrance to the pipe is turbulent, the flow will remain purely laminar if the pipe is straight, reasonably smooth and sufficiently long.

For laminar flow, the entrance length, L, is a function of the Reynolds number thus:

$$\frac{L}{D} \cong (0.06)\frac{\rho v D}{\mu} \tag{111}$$

Where D is the diameter of the pipe,

ρ is the density of the fluid,

v is the average flow velocity

and μ is the viscosity of the fluid.

Thus from the equation (4.1), it follows that

$$\frac{L}{D} \cong (0.06)\mathrm{Re}, \qquad since \ \mathrm{Re} = \frac{\rho v D}{\mu}$$

Implying that $L \cong (0.06)\mathrm{Re}\, D$ (112)

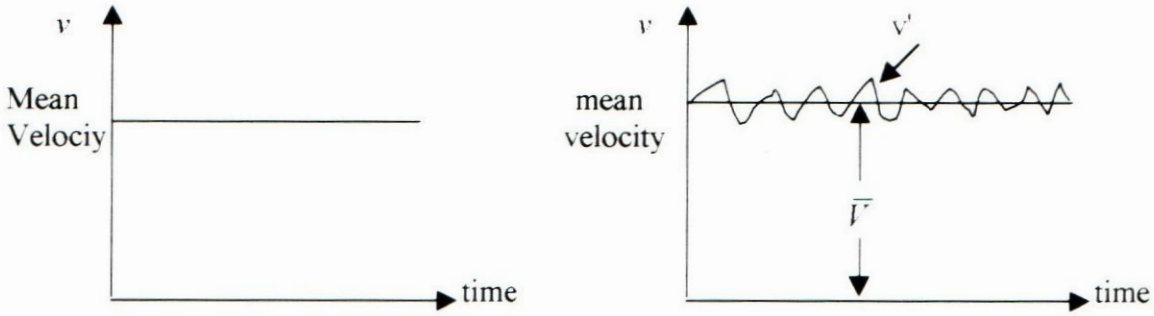
Laminar flow in a pipe is however expected only for Reynolds number (Re) less than about 2300. Therefore the entrance length (L) for laminar pipe flow may be as long as

$$L \cong 0.06\,\mathrm{Re}\, D \le (0.06)(2300)D \le 138D \tag{113}$$

One can obtain a more quantitative picture of the difference between laminar and turbulent flow by examining the output from a sensitive velocity measuring device immersed in the flow. If one measures the x component of velocity at a fixed location in a pipe for both laminar and turbulent steady flow, the traces of velocity versus time would be as shown in the figure 4.2. For steady laminar flow, the velocity at a point remains constant with time. In turbulent flow the velocity trace indicates a random fluctuation of the instantaneous velocity, v, about the time mean velocity $\bar{v}$. Thus we consider the instantaneous velocity, U, as the sum of time mean velocity, $\bar{v}$ and the fluctuating component v', that is

$$v = \bar{v} + v'$$

Because the flow is steady, the mean velocity, $\bar{v}$, does not vary with time.



(a) laminar flow (b) turbulent flow

Figure 4.2 Mean velocity verses time for laminar and turbulent flows

Although many turbulent flows of interest are steady in the mean (i.e. $\bar{v}$ is not a function of time), the presence of the random, high frequency velocity functions make the analysis of turbulent flows extremely difficult. In a one dimensional laminar flow, the shear stress is related to velocity gradient by the simple relation:

$$\tau_{xx} = \mu \frac{du}{dy} \tag{114}$$

For turbulent flow in which the mean velocity field is one dimensional, no such simple relation is valid. Random, three-dimensional velocity functions ($u', v' and w'$) transport momentum across the mean flow streamlines, increasing the effective shear stress. In turbulent flow, there is no universal relationship between the stress field and the mean velocity field. Thus in turbulent flows, we must rely heavily on semi-empirical theories and on experimental data.

4.3 Fully Developed Laminar Flow in a Pipe

There are relatively few viscous flow problems for which we can obtain analytical solutions in closed form, but the method of solution is important. We shall consider a few classic examples of fully developed laminar flow. Our interest is in obtaining some detailed information about the velocity field, which permits calculations of shear stress, pressure drop and flow rate in which we shall relate with the dimensionless parameter; the Reynolds number (Re).

We shall use the Navier-Stokes equations and the mass conservation equation to find a complete solution for some simple geometries.

4.3.1 Flow between Two Parallel Infinite Plates - Plane Poiseuille Flow (Both plates being stationary)

In plane Poiseuille flow, $\frac{dp}{dx} \neq 0$ i.e. influence of pressure gradient is felt and both the plates are stationary.

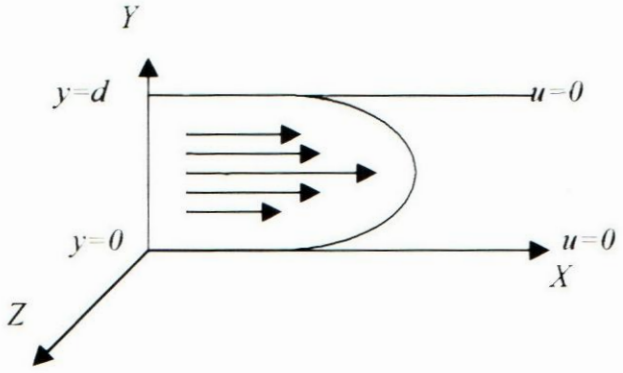


Figure 4.3 Flow through two parallel stationary plates

Here, no slip conditions demand that:

$u = 0$ at $y = 0$

$u = 0$ at $y = d$

using the boundary conditions the equation for velocity distribution i.e

$$u = Ay + B - \frac{1}{2\mu} py^2 \tag{115}$$

becomes

$$u = \frac{p}{2\mu}(d - y)y \tag{116}$$

The skin friction (shear stress) thus becomes

$$\tau = \frac{d}{dy}\left[\frac{p}{2\mu}(d - y)y\right]$$

$$= \frac{pd}{2} - py$$

at $y=0$, the $\tau = \frac{1}{2} pd$ (117)

and

at $y = d$, $\tau = \frac{-1}{2} pd$ (118)

Thus the skin-friction is the same at both the plates except for the sign.

4.3.2 Flow Through a Circular Pipe

We consider an infinite pipe of radius a. This is another situation where an exact solution is possible and its simple in nature, because the non-linear terms drop. This flow is also called Hagen-Poiseuille flow.

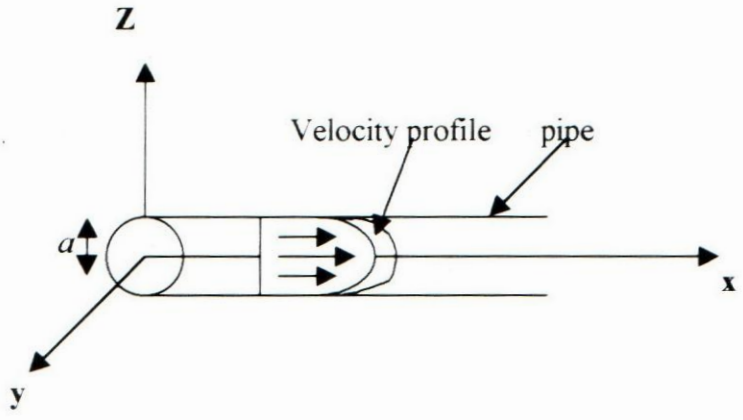


Figure 4.4 Velocity profile for flow through circular pipe

Here the flow is axisymetric, (i.e. along the axis of the pipe) and depends only on radial distance from the axis. In this case we take the basic equation in cylindrical polar co-ordinates (r, θ, z) then the velocity vector can be taken as $\vec{q} = (0,0,u_z(r))$.

The momentum equations are

$$0 = \frac{-\hat{c}p}{\hat{c}r} \tag{119}$$

$$\left.\begin{aligned} 0 &= \frac{-\partial p}{\partial \theta} \\ 0 &= \frac{-\partial p}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) \end{aligned}\right\} \qquad (120)$$

Continuity equations becomes

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

which is satisfied.

Equations (119) and (120) shows that, the pressure does not depend upon r and θ.

Moreover by differentiating (120) with respect to z we find that

$$\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial z}\right) = 0$$

which implies that

$$\frac{\partial p}{\partial z} = -p \qquad (constant)$$

Thus equation (120b) becomes

$$\frac{d^2 u_z}{dr^2} + \frac{1}{r}\frac{du_z}{dr} + \frac{1}{\mu}p = 0 \qquad (121)$$

Integrating equation (121) yields

$$u = A\log r + B - \frac{p}{4\mu}r^2 \qquad (122)$$

Where A and B are arbitrary constants to be determined from the boundary conditions. Since at the axis of the pipe the velocity should be finite, therefore A should be zero, that's

$$U_{max} = A\log 0 - B$$

$$U_{max} = A\infty - B$$

which is a contradiction implying that $A = 0$.

Moreover, no-slip condition gives that $u=0$ at $r = a$. Thus equation (122) becomes

$$u = \frac{-p}{4\mu}(r^2 - a^2) \tag{123}$$

hence the skin-friction /shear stress becomes

$$\tau = \mu \frac{du}{dy}_{r=a} = \frac{-1}{2} pa \tag{124}$$

Volumetric flow rate becomes

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$$Q = \int_0^a \frac{p}{4\mu}(r^2 - a^2) 2\pi r dr$$

$$= \frac{-\pi a^4 p}{8\mu} \tag{125}$$

Therefore the flow rate as a function of pressure drop will be

$$Q = \frac{\pi a^4 \Delta p}{8 \mu L} \tag{126}$$

but $a = \frac{D}{2}$ where D is the diameter of the pipe. Therefore Equation (4.16) becomes

$$Q = \frac{\pi \Delta p D^4}{128 \mu L} \tag{127}$$

for laminar flow in a horizontal pipe.

The average velocity $\bar{v}$ is then given by

$$\bar{v} = \frac{Q}{A} = \frac{Q}{\pi a^2}$$

$$= \frac{-a^2 p}{8\mu} \tag{128}$$

To find the point of maximum velocity, we set $\frac{\partial u}{\partial r} = 0$ and solve for the corresponding value of r.

From the equation (122),

$$\frac{du}{dr} = \frac{pr}{2\mu}$$

thus $\frac{du}{dr} = 0$ at $r = 0$

$$\text{At } r = 0, \quad u = u_{max} = \frac{-a^2 p}{4\mu} = 2\bar{v} \tag{129}$$

The parabolic velocity profile, given by equation (122) for fully developed laminar flow

in a pipe is sketched as shown by figure 4.5 below.

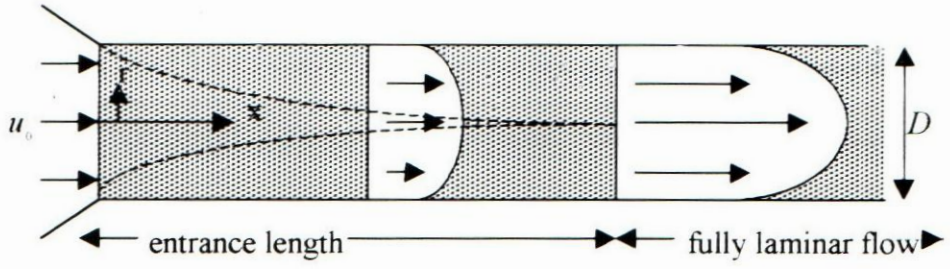


Figure 4.5 Parabolic velocity profile for fully Laminar flow

4.4 Calculation of Head Loss in a Pipe Flow

Total head loss, h_{LT}, is regarded as the sum of major losses, h_L, due to frictional effects in a fully developed flow in constant area of tubes, and minor losses, h_{LM}, due to entrances, fittings, area changes etc.

Considering major losses, (friction factor), the energy balance equation (Bernoulli's equation) expressed as

$$\left(\frac{p_1}{\rho}+\frac{\alpha_1 v_1}{2}+gz_1\right)-\left(\frac{p_2}{\rho}+\frac{\alpha_2 v_2}{2}+gz_2\right)=h_{lT} \tag{130}$$

Can be used to evaluate the major head loss. For fully developed laminar flow through a constant area pipe; minor head losses (h_{LM}) is zero,

$$h_{LM}=0 \text{ and } \alpha\frac{v_1^2}{2}=\frac{\alpha_2 v_2^2}{2}$$

hence equation (130) reduces to

$$\frac{p_1-p_2}{\rho}=g(z_2-z_1)+h_l \tag{131}$$

if the pipe is horizontal, then $Z_2 = Z_1$ and hence

$$\frac{\Delta p}{\rho} = h_L \tag{132}$$

Thus major head losses can be expressed as the pressure loss for fully developed laminar through a horizontal pipe of constant area. Since head loss represents the energy converted by frictional effects from mechanical to thermal energy, head loss for fully developed laminar flow in a constant area ducts depends only on the details of the flow through the duct.

Considering laminar flow, the pressure drop may be expressed analytical for fully developed flow in a horizontal pipe as

$$\Delta p = \frac{128\mu LQ}{\pi D^4} = \frac{32L\mu v}{D^2} \tag{133}$$

since $Q = vA$

Where A is the cross sectional area given by $A = \pi D^2 / 4$.

Substituting equation (133) into equation (132) we obtain

$$h_l = \frac{64}{\mathrm{Re}} \frac{Lv^2}{2D} \tag{134}$$

Where $Re = \frac{\rho vD}{\mu}$ is the Reynold's number. This shows that Reynolds number affects the head losses in the flow, since we can express the major head loss (h_L) in terms of the dimensionless parameter Re.

In turbulent flow, pressure drop cannot be evaluated analytically, hence experimental results and dimensional analysis is used as an aid in correlating the experimental data. In a fully developed turbulent flow, the pressure drop, Δp due to friction in a horizontal constant area pipe is known to depend on the pipe diameter, D, the pipe length, L, the

pipe roughness, e, the average velocity v, the fluid density, ρ, and the fluid viscosity, μ, in functional form as

$$\Delta p = \Delta p(D, L, e, v, \rho, \mu) \tag{135}$$

Applying dimensional analysis to equation (135) we obtain

$$\frac{\Delta p}{\rho v^2} = f\left(\frac{\mu}{\rho v D}, \frac{L}{D}, \frac{e}{D}\right) \tag{136}$$

we recognize that $\frac{\mu}{\rho v D} = \frac{1}{\mathrm{Re}}$, hence equation (136) becomes

$$\frac{\Delta p}{\rho v^2} = \phi\left(\mathrm{Re}, \frac{L}{D}, \frac{e}{D}\right) \tag{137}$$

substituting from equation (133) we obtain

$$\frac{h_f}{v^2} = \frac{L}{D}\phi_1\left(\mathrm{Re}, \frac{e}{D}\right) \tag{138}$$

since ϕ_1 is undetermined, it is permissible to introduce a constant into the left side of the above equation (138). The number $1/2$ is introduced into the denominator such that the head loss is non- dimensionalized on the kinetic energy per unit mass of flow. Then

$$\frac{h_f}{\frac{1}{2}v^2} = \frac{L}{D}\phi_2\left(\mathrm{Re}, \frac{e}{D}\right) \tag{139}$$

The unknown function $\phi_2\left(\mathrm{Re}, \frac{e}{D}\right)$ is defined as the friction factor,

$$\mathrm{f} = \phi_2\left(\mathrm{Re}, \frac{e}{D}\right) \tag{140}$$

and thus

$$h_f = f\frac{L}{D}\frac{v^2}{2} \tag{141}$$

This equation (140) and (141) shows that, the major head loss is affected by the Reynolds number (*Re*).

The friction factor *(f)* for laminar flow may therefore be obtained by comparing equation (134) and (141). Thus

$$h_f = \left(\frac{64}{\mathrm{Re}}\right)\frac{L}{D}\frac{v^2}{2} = f\frac{L}{D}\frac{v^2}{2}$$

Consequently, for laminar flow

$$f_{laminar} = \frac{64}{\mathrm{Re}} \qquad (142)$$

Which implies that, friction factor for laminar flow is a function of Reynolds number only. It is independent of roughness.

Varying the average flow velocity may change Reynolds number in a pipe most easily. If the flow in a pipe is originally laminar, increasing the velocity until the critical Reynolds number is reached causes transition to occur; the laminar flow gives way to turbulent flow. This change in velocity profile causes the wall shear stress to increase sharply, with the same effect on the friction factor.

As the Reynolds number is increased above the transition value, the velocity profile continues to become fuller as shown by figure 4.6 below

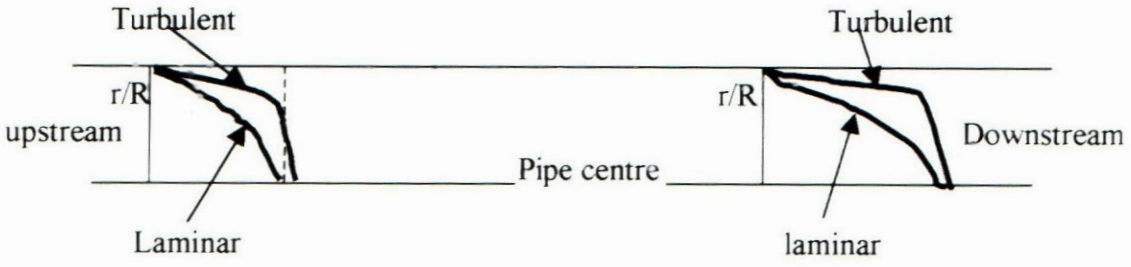


Figure 4.6 :Velocity profile for both laminar and turbulent flow regimes for the same Reynolds number in a smooth circular pipe

The friction factor at first tends to follow the smooth pipe curve, along which friction factor is a function of Reynolds number *(Re)* only. However, as the Reynolds number increases, the velocity profile becomes still fuller. The size of the thin "Viscous sub layer" near the tube wall decreases. As roughness elements begin to poke through this layer, the effect of roughness becomes important and the friction factor becomes a function of the Reynolds number (*Re*) and the relative roughness.

At very large Reynolds number, most of the roughness elements on the tube wall protrude through the viscous sublayer; the drag and hence the pressure loss, depend only on the size of the roughness elements. This flow regime is termed as the "fully-rough" flow regime; friction factor depends on $\frac{e}{D}$ only.

4.5 Conclusion

In conclusion we see that an increase in Reynolds number causes a decrease in friction as long as the flow remains laminar, this is a very useful aspect in the study of flows in pipes. At transition, friction factor (f) increases sharply. In the turbulent flow regime, the friction factor decreases gradually along the smooth pipe curve, and finally levels out at a constant value for extremely large Reynolds number. Thus we can see that knowledge of Reynolds number and friction factor (f) for laminar and turbulent pipe

flows is of great importance in piping systems. Values of relative roughness ($\frac{e}{D}$) and friction factor (f) which is a function of Reynolds number (Re) helps to tell the effectiveness of the piping systems with time due to the fact that corrosion can weaken piping systems, eventually leading to failure of the system.

CHAPTER FIVE

DRAG, LIFT, BOUNDARY LAYER SEPARATION AND BOUNDARY LAYER CONTROL

5.1 Introduction

Any object moving through a fluid will experience a drag force, that's a net force in the direction of flow due to the pressure and shear forces on the surface of the object. Most of the information pertaining to the drag force on objects is a result of many experiments with wind tunnels, water tunnels, and other ingenious devices that are used to measure the drag on scale models.

Typically, the drag coefficient is given as

$$C_D = \frac{2D}{\rho u^2 A} \tag{143}$$

where D is the drag force

and C_D is a function of other dimensionless parameters such as the Reynolds number, Re, Mach number ,Ma, Froude number, Fr, and relative roughness of the surface, $\frac{e}{l}$,

$$\text{that's } C_D = f \text{ (shape, } Re, Ma, Fr, \frac{e}{l}) \tag{144}$$

When an object also moves through a fluid, there's a force normal to the free stream if the object is not symmetrical such that it doesn't produce a symmetric flow field. This force is called the lift force. Considerable effort has been put forth to understand the various properties of the generation of this lift force. Objects have been designed to reduce this lift force so as to suit their use, others have been

designed to increase this lift force so as to serve their purpose well. Typically the lift force is given in terms of the lift coefficient C_L as

$$C_L = \frac{2L}{\rho u^2 A} \tag{145}$$

where L is the lift force.

Similarly, the lift coefficient is a function of some appropriate dimensionless parameters and as the drag coefficient,

$$C_L = f \text{ (Shape, } Re, Ma, Fr, \frac{e}{l}\text{).} \tag{146}$$

In this chapter, we shall look at the character of C_D and C_L as a function of the above parameters, however, emphasis shall be placed on the Reynolds number effect and their relative applications in our daily living. We shall aim at relating the boundary layer analysis to the drag and lift force with respect to the Reynolds number.

5.2 The Drag Force

5.2.1 Friction Drag

Friction drag, D_f is the part of the drag that's due directly to the shear stress on the object. It is a function of not only the magnitude of the wall shear stress, but also of the orientation of the surface on which it acts. If the surface is parallel to the upstream velocity, the entire shear force contributes directly to the drag. This is true for the flat plate parallel to the flow. If the surface is perpendicular to the upstream velocity, the shear stress contributes nothing to the drag.

In general, the surface of a body will contain a portion parallel to and normal to the upstream flow, as well as any direction in-between. A circular cylinder is an example of such a body. Since the viscosity of most common fluids is small, the contribution of the shear force to the overall drag on a body is often quite small. That's because Reynolds number of familiar flows are quite large, the percentage of the drag caused directly by shear stress is often quite small. For highly streamlined bodies or for low Reynolds number flows, most of the drag may be due to the friction drag.

The friction drag on a flat plate of width b and length l oriented parallel to the upstream flow can be calculated from

$$D_f = \frac{1}{2}\rho u^2 blC_{Df} \tag{147}$$

Where C_{Df} is the friction drag coefficient. The value of C_{Df}, given as a function of Reynolds number $\mathrm{Re}_l = \frac{\rho u l}{\mu}$ and relative surface roughness, $\frac{e}{l}$, is as shown by figure 5.1 and table 5.1 below

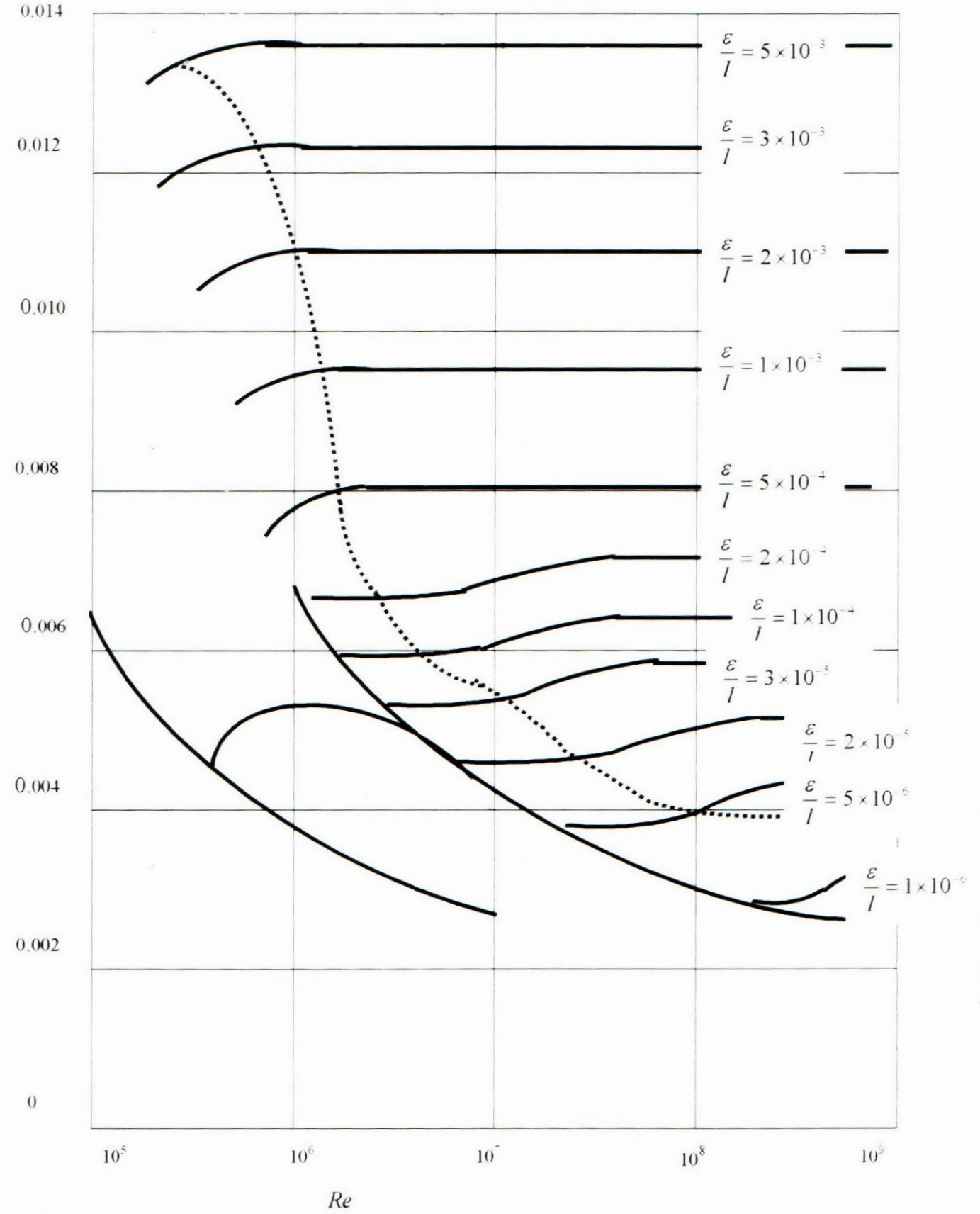


Figure 5.1 : Friction Drag coefficient for a flat plate parallel to the upstream flow

Equation	Flow conditions
$C_{Df} = \frac{1.328}{(\mathrm{Re})^{0.5}}$	Laminar flow
$C_{Df} = \frac{0.455}{(\log \mathrm{Re})^{2.58}} - \frac{1700}{\mathrm{Re}}$	Transitional with $\mathrm{Re}_{cr} = 5 \times 10^5$
$C_{Df} = \frac{0.455}{(\log \mathrm{Re})^{2.58}}$	Turbulent, smooth plate
$C_{Df} = \left[1.89 - 1.62 \log\left(\frac{e}{l}\right)\right]^{-2.5}$	Completely turbulent

Table 5.1 Friction drag coefficient and the flow conditions

Note that from figure 5.1 and table 5.1, the friction drag coefficient C_{Df} for most of the flow conditions is a function of Reynolds number. We therefore see the significance attached to the Reynolds number when it comes to the study of friction drag coefficient in moving objects.

5.2.2 Pressure Drag

This is that part of the drag that's due directly to the pressure, p, on the object. It's often referred to as the form drag because of its strong dependency on the shape or form of the object. Pressure drag, D_p is a function of the magnitude of the pressure and the orientation of the surface element on which the pressure force acts.

For flows in which viscous effects are large relative to the inertia effects, both the pressure difference and the wall shear stress scale with the characteristic viscous

stress, $\frac{\mu U}{l}$, where l, is the characteristic length. In such situations, the drag coefficient is expected to be proportional to $1/Re$. That is

$$C_D \approx \frac{D}{\left(\frac{\rho u^2}{l}\right)} \approx \frac{\left(\frac{\mu u}{l}\right)}{\left(\frac{\rho u^2}{2}\right)} \approx \frac{\mu}{\rho u l} = \frac{1}{\mathrm{Re}}.$$

These characteristics are similar to the friction factor, f, dependence of $f \sim 1/Re$ for laminar pipe flow and $f \sim$ constant for large Reynolds number flow.

If the viscosity were zero, the pressure drag on any shaped object in a steady flow would be zero. There perhaps would be a large pressure forces on the front portion of the object, but there would be equally large pressure forces on the rear portion. If the viscosity is not zero, the net pressure drag may be non-zero because of boundary layer separation.

5.3 Boundary Layer Separation and Vortex Formation

Considering the effects of large Reynolds number as discussed in chapter three, it's very important to note that the nature of velocity profile in the boundary layer close to the wall depends on whether the potential velocity outside the boundary layer is accelerating or decelerating. In accelerating flow, $\frac{dp}{dx} < 0$,and in decelerating flow , $\frac{dp}{dx} > 0$. For flow with adverse pressure gradient, sometimes there's a reversal in the direction of flow near the vicinity of the wall. When such a reversal flow takes place, then we say that the boundary layer has separated from the wall.

Illustration.

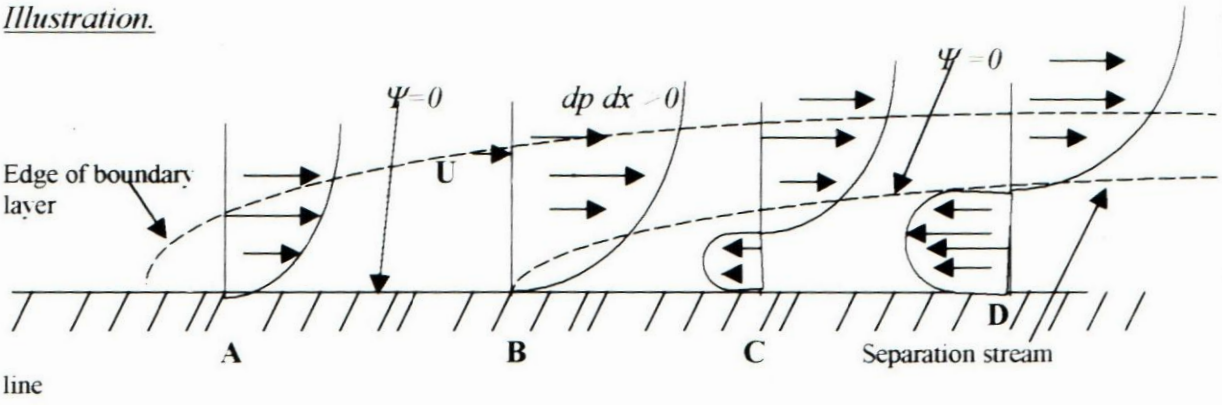


Figure 5.2 Flow over a flat plate with adverse pressure gradient

At point A, velocity gradient $\frac{\partial u}{\partial y} > 0$; at point C, $\frac{\partial u}{\partial y} < 0$. Separation of the boundary layer from the wall starts at B where $\frac{\partial u}{\partial y} = 0$. Beyond the point B in the downstream direction, reversed flow induces eddy formation. The streamline of zero velocity coinciding with the boundary wall upstream of the separation point, leaves the surface at point B. A new boundary is formed for the regular mainstream flow after separation of the boundary layer. Below the separation streamline, the region characterised by irregular turbulent eddies is called the wake.

Although separation of the boundary layer cannot occur without adverse pressure gradient, there can be adverse pressure gradient and no separation. Separation can occur in a diffuser or in a highly divergent channel. For flow over a flat plate with zero pressure gradient, backward flow is never possible and so boundary layer separation doesn't occur at all. Boundary layer separation can occur either on a straight wall or on a curved wall, with flow having adverse pressure gradient.

The mechanism of boundary layer separation can be explained further by considering the flow of a real fluid around a circular cylinder as shown by figure 5.3 below.

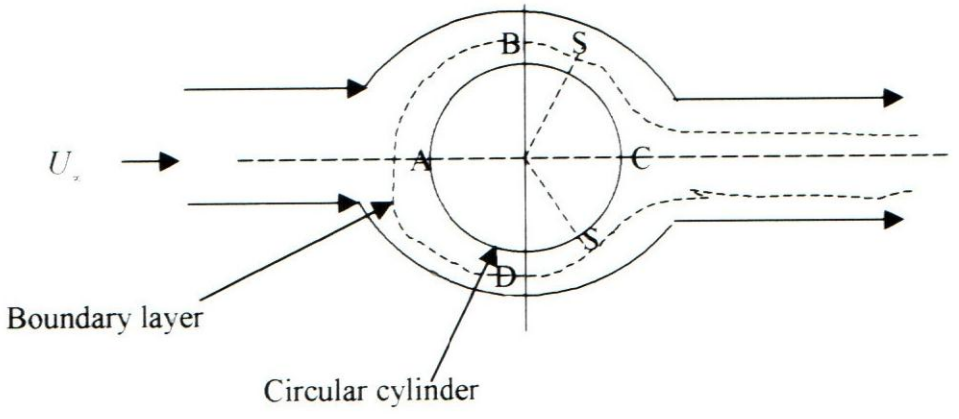


Figure 5.3 Separation of boundary layer and vortex formation around a circular cylinder.

In region *AB*, pressure within the boundary layer decreases and the potential velocity increases in the downstream direction. Consequently there is sufficient kinetic energy available in the boundary layer to overcome the frictional resistance in the region. But the situation changes beyond the point *B*. If the boundary layer were to continue in the entire region *BC*, a part of the kinetic energy in the boundary layer has to be spent to overcome frictional resistance and a fraction of this energy has to be converted into pressure. But soon kinetic energy would decrease, and hence the fluid particles in the neighbourhood of the wall come practically to rest in the vicinity of the point *S*. Fluid particles in the upper layer, however still posses sufficient kinetic energy and try to move downstream towards the point *C*, as in

potential flow. Beyond the point S, therefore backward flow in the vicinity of the wall due to somewhat high pressure in front, and forward flow in the upper layer cause vortex formation. The boundary layer detaching itself from the wall at S takes the path where the kinetic energy loss minimises. The pressure in the wake is close to the pressure of the separation point since the turbulent eddies cannot convert its rotational kinetic energy into pressure head. Since pressure in the wake is less than at the forward stagnation point A as shown by the figure 5.3 above, the cylinder experiences a drag force because of net pressure difference. The drag force is the form drag, which is created by the resultant of the normal forces whereas the resultant of tangential forces gives rise to the skin friction drag. The sum of the two drag forces is called the profile drag.

The wake behind a circular cylinder has been investigated both analytically and experimentally. In the range $60 < Re < 5x10^5$, eddies are shed alternately from the top and bottom of the cylinder and move downstream as shown by figure 5.4 below

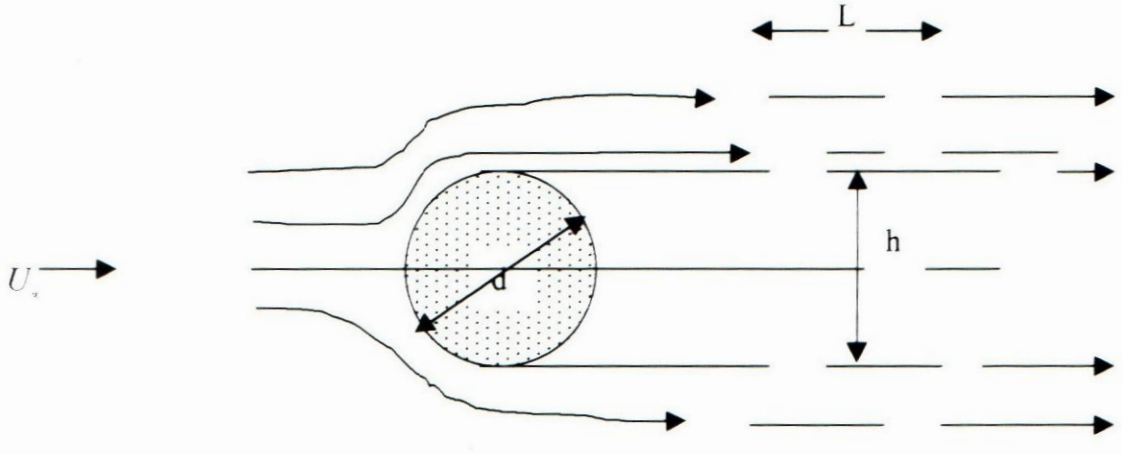


Figure 5.4 Karman vortex street behind a cylinder

The regular pattern of these alternately clockwise and counterclockwise vortices in the wake called Karman Vortex Street. According to T Von Karman, these vortices are stable with respect small disturbances only if $\frac{h}{l}$=0.28

The frequency f of shedding the vortices can be expressed as a dimensionless parameter $\frac{fd}{U_\infty}$, known as the Strouhal number, which is a function of Reynolds number, by the empirical relation

$$\frac{fd}{U_\infty} = 0.198\left(1-\frac{19.7}{\mathrm{Re}}\right) for 250 < \mathrm{Re} < 2\times 10^5 \qquad (148)$$

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At high Reynolds number, the Strouhal number approaches a constant value of *0.198*. The shedding of vortices by turns causes periodic lateral forces, which may cause forced vibration of the cylinder. This accounts for the "singing" of telephone and power transmission lines in the wind when the frequency of vortex shedding is nearly equal to the natural frequency of the wires. The aerodynamic vibration caused by shedding of vortices may sometimes prove to be dangerous to structures such as chimneys, submarines Periscopes, suspension bridges, and aircraft wings because it may cause breakage of these structures.

Separation can occur both in laminar and turbulent flows. However, a turbulent boundary layer can offer greater resistance to separation than a laminar layer. The reason for this is turbulence transforms kinetic energy from upper layers to the layers adjacent to the wall through the fluctuating transverse velocity components. Consequently the separation point moves further down stream in turbulent flow

against adverse pressure gradient. This explains why the width of the wake behind a cylinder is narrower for turbulent flow.

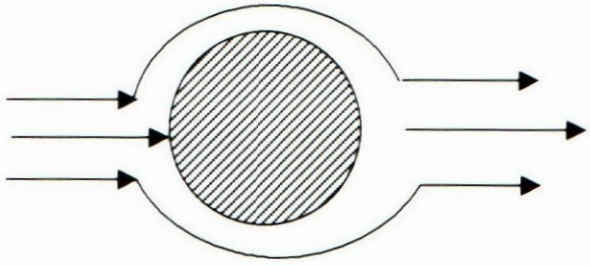


Figure.5.5 (a) No separation

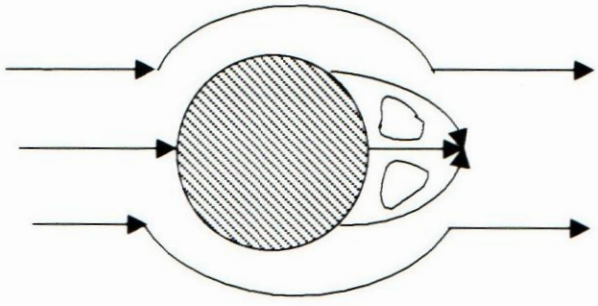


Figure 5.5(b) Steady separation bubble

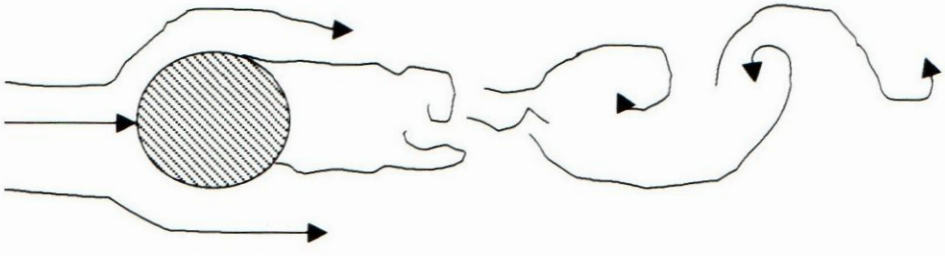


Figure 5.5(c) Osculating Karman vortex street

point of separation on the rear end. A rounded nose at the front stagnation point is desirable for subsonic flow to minimise disturbances of streamlines.

Fluid particles close to the wall lose their kinetic energy near the separation point. Separation can be prevented if these particles of reduced kinetic energy can be removed from the boundary layer and replaced by fluid particles through a series of slots in the wall. The particles of higher kinetic energy in the upper layers now move close to the wall and can sustain the boundary layer further down stream. Moreover, suction makes the boundary layer thinner and transition form laminar to turbulent boundary layer is delayed, thereby reducing the skin friction drag. This method is sometimes used to prevent separation of boundary layer around aircraft wings.

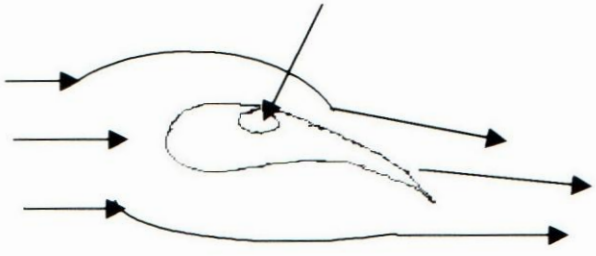


Figure 5.6 Separation control by suction

Separation can also be inhibited when a fluid is injected into the boundary layer for mixing with the layer and for shifting it away from the wall. The high energy fluid particles should be blown tangentially from a backward facing slot, so that it can

merge with the main stream smoothly. Injection of fluid provokes turbulence, which increase the skin friction. Thus, injection of fluid in the boundary layer is more suitable for suppressing an immediate separation of a turbulent boundary layer.

5.4 Lift Force on Immersed Bodies in Motion

As stated in the introductory part of this chapter, the lift coefficient C_L is defined as

$$C_L = \frac{2L}{\rho u^2 A} \tag{149}$$

where L is the lift force.

This coefficient is a function of some appropriate dimensionless parameters , that is

$$C_L = \phi \text{ (shape, } Re, Ma, Fr \frac{e}{l}). \tag{150}$$

The Froude number (Fr) is important only if there is a free surface present, as with an underwater "wing" used to support a high speed hydrofoil surface ship. After the surface roughness , $\frac{e}{l}$ is relatively unimportant in terms of lift . It has more of an effect on the drag. The Mach number, ma is of importance for relatively high – speed subsonic and supersonic flows; and the Reynolds number effect is often not great. The most important parameter that affect the lift coefficient is the shape of the object. Considerable effort has gone into designing optimally shaped lift – producing devices (e.g. airfoils, fans, spoilers on cars, etc) operate in the large Reynolds number range. In which the flow has a boundary layer character, with viscous effects confined to the boundary layers and wake regions. For such cases the wall shear stress contributes little to the lift.

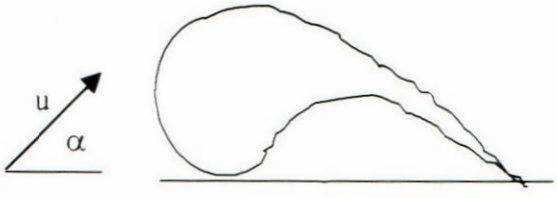
For objects operating in very low Reynolds number regimes ($Re < 1$), Viscous effects are important, and the contribution of the shear stress to the lift may be as important as that of the pressure. Such situations include the flight of minute insects and the swimming of microscopic organisms.

A typical device designed to produce lift does so by generating a pressure distribution that is different on the top and bottom surfaces for large Reynolds number flows these pressure distribution are usually directly proportional to the dynamic pressure, $\frac{\rho u^2}{2}$, with viscous effects being of secondary importance.

Consider the two airfoils used to produce lift indicated by figure 5.7 below. Clearly the symmetrical one cannot produce lift unless the angle of attack, α , is now zero. Because of the asymmetry of the non-symmetric airfoil, the pressure distributions on the upper and lower surfaces are different, and a lift is produced even with $\alpha = 0$



Symmetrical



Nonsymmetrical

Figure. 5.7 symmetry of an airfoil and the angle of attack (α)

In general, the lift coefficient increases and the drag coefficient decreases, with an increase in aspect ratio. Although viscous effects and the wall shear stress contribute little to the direct generation of lift they play an extremely important role in the design and use of lifting devices. This is because of the viscosity – induced boundary layer separation that can occur on non-streamlined bodies such as airfoils that have too large an angle of attack. The lift coefficient increases rather steadily with the angle of attack. If α is too large, the boundary layer on the upper surface separates, the flow over the wing develops a wide, turbulent wake region, the lift decreases, and the drag increases. The airfoil stalls. Such conditions are extremely dangerous if they occur while the aeroplane is flying at a low altitude where there is no sufficient time and altitude to recover from the stall.

Therefore in conclusion, the Reynolds number effects on fluid flows is of great importance in the sense that, large Reynolds number flows brings about boundary layer flows from which the drag and lift on bodies can be determined hence, more information on flying objects can be obtained. However, the shape and other non-dimensional parameters like the Froude number, and the Mach number also plays a role in the determination of the drag coefficient C_D and the lift coefficient C_L acting on these objects.

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