"SOME ASPECTS OF
NATURAL CONVECTION BOUNDARY LAYER FLOWS"

BY

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DECLARATION

THIS PROJECT IS MY ORIGINAL WORK AND HAS NOT BEEN PRESENTED IN ANY OTHER UNIVERSITY FOR ANY OTHER DEGREE.

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I am also greatly indebted to Prof V.P. Onyango, Chairman Mathematics Department for his encouragement and advice in undertaking the research and writing this final report. Much thanks go to Prof W.K. Kipng'eno, Dean Faculty of Science and Prof S. Katia, director of Graduate school for their encouragement and administrative guidance. May I also thank,Dr.Sogomo, Mr. Gathogo, Dr. Thorofo and Prof Singh all of Mathematics Department, Egerton University.

May I express my sincere thanks to all my colleagues especially John Lonyangapuo, friends and well wishers who have lent their unstinted cooperation and valuable suggestion to complete my project. Special thanks go to my husband Joseph Kiarie Ranji and son Kevin, parents, sisters and brothers for their constant encouragement and tolerance.

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PREFACE

Fluid mechanics is not a subject to be studied for theoretical purposes only but its of much use in engineering and technology.

Chapter one is introduction to the various specialized vocabulary of fluid mechanics. This helps the reader to become aware of the various applications of fluid mechanics in order to familiarize himself to the following chapters.

Chapter two gives an account of the work which has been done by several researchers on natural convection boundary layer flows.

In chapter three two basic concepts of heat and heat transfer, have been given. Importance of heat transfer, modes of heat transfer and material properties of importance in heat transfer have also been given. Simplified form of the Navier-stokes equation have been presented. With special attention on natural convection heat transfer,we have come up with the exact and approximate methods of a flat plate at zero incidence.

Chapter four contains a research paper which is my original work on free convective flow through porous medium. Analytical expressions for the velocity and the temperature fields have been given. The effects of Grashof number (G_r) and the permeability parameter (**K**) on the velocity field have been shown graphically.

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CHAPTER ONE

1.0 INTRODUCTION:

1.1 Fluid:

Most materials are classified into solids, liquids and gases. The properties of each group varying from those of the other though some possess dual properties.

A solid has definite shape and liquids maintain a fixed volume. Liquids and gases take the shape of the container in which they are put but liquids do not change with the size of the container. Volume of gases change with pressure or temperature changes and are compressible but liquids are incompressible and their volume does not change with pressure or temperature change.Materials undergo deformation but the magnitude of deformation varies. Fluids are highly deformed even with small external shear forces but solids are slightly deformed even for large external shear forces.

Then a fluid can be defined as a substance that deforms continuously under the application of a shear stress no matter how small the shear stress may be. A fluid cannot sustain a shear stress when at rest because the fluid motion continues under the application of a shear stress.

Fluids are made up of liquids and gases. A fixed amount of liquid has a definite volume which varies slightly with temperature and pressure. If the containing vessel has a greater capacity than this volume, the liquid occupies only part of this container and forms an interface separating it from its own vapour, the atmosphere and other gases present. For a fixed amount of gas in a closed container, it expands until its volume equals that of the container. Under normal conditions, liquids are incompressible but gases can be compressed readily.When a gas undergoes a negligible change of volume, its behaviour is similar to that of a liquid and incompressible. When there is a big change in volume, the compressibility of the gas is considered when examining its behaviour.

A fluid can be treated as a continuum, that is, a continuous distribution of matter with no empty space. We shall deal mainly with the smallest volume of fluid where macroscopic properties like density,pressure and temperature have their usual meaning and motion of the fluid in contact with the solid surface is similar to that of the surface.

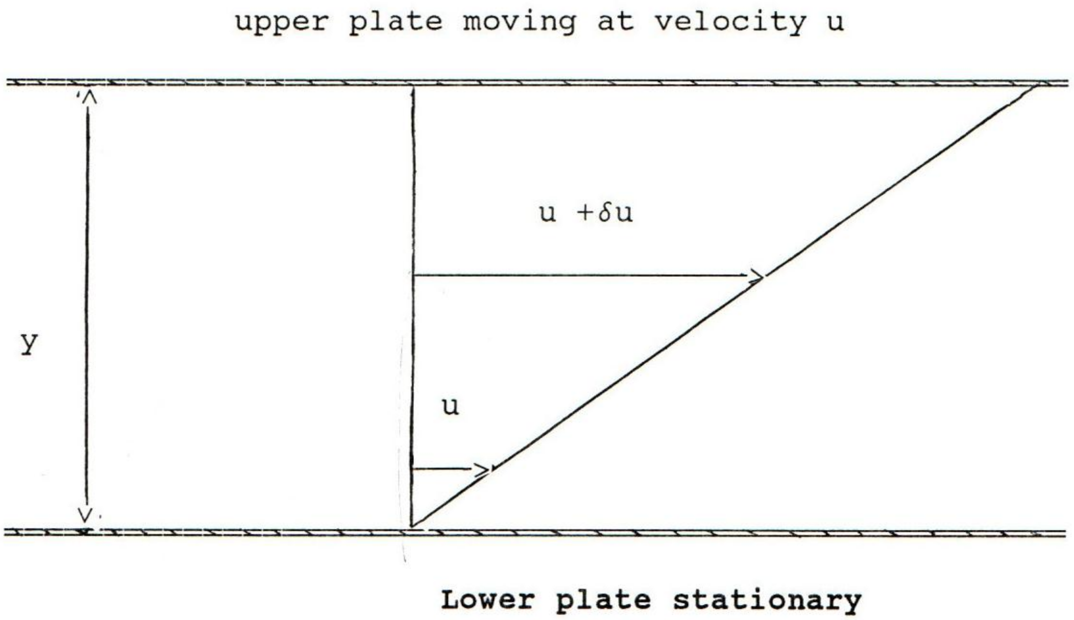
1.2 Fluid properties:

Real fluids possess some properties which are important in science. For real fluids in motion viscosity is very important. For fluids at rest, compressibility, vapour pressure, fluid density ,capillarity and surface tension are of great importance.

In this project we shall deal mainly with thermal properties which compose of gas constant,enthalpy and entropy for gases. For both liquids and gases we have conductivity, internal energy and specific heat capacities. Other fluid properties arise from the above properties like fluid viscosity and thermal diffusivity arise from thermal conductivity and density.Kinematic viscosity involves thermal diffusivity

1.2.1 Viscosity

From above discussion, a fluid was defined as a substance that deforms continuously under the action of a shear stress. With no shear stress, there is no deformation. Fluids can be classified according to the relation between the applied shear stress and the rate of deformation. Viscosity is a property of a fluid that determines the amount of this resistance.For mathematical understanding we consider a fluid confined between two parallel plates where one is at rest and the other moving with a constant velocity parallel to itself as shown in the figure below.



$\mu = \frac{du}{dy}$

Fig 1

If the pressure remains constant, then the fluid adheres to both walls so that velocity at the lower plate is zero and at the upper plate is equal to the velocity of the plate, u.

Rate of change of velocity $=(u+\delta u)-u\ /\delta y$

$$=\delta u/\delta y = du/dy$$

If the velocity distribution in the fluid between the plates is linear, and if the shearing stress is assumed to be Proportional to the rate of velocity, then

$$\tau \alpha du/dy$$

$$\tau = \mu du/dy \text{ ------------------} (1)$$

where τ = unit shearing stress

μ = proportionality constant

du/dy =velocity gradient

τ depends on the nature of the fluid, it is small for fluids like water or alcohol but large for very viscous fluids like oil or glycerine.

μ mainly depends on the temperature of the fluid and is called viscosity.

Equation (1) gives the Newton's law of friction and can be rearranged as

$$\mu = \tau/(du/dy)$$

To get the dimensions of viscosity, $\mu = N/m^2/(m/s)/m = Ns/(m^2)$

For fluid motions in which frictional and inertia forces interact, we have kinematic viscosity which is given by the ratio of viscosity μ to density ρ and denoted by the greek letter ν that is,

$$\nu = \mu/\rho$$

Its international system is given by

$$\nu = \mu/\rho = Ns/m^2/kg/m^3 = m^2/s$$

Kinematics viscosity ν for liquids has the same type of temperature dependence as μ because density ρ changes slightly with temperature. For gases, ρ decreases with increasing temperature and ν increases rapidly with temperature.

1.3 Types of fluids

An ideal fluid is one which is incompressible and has zero viscosity. With zero viscosity, the fluid offers no resistance to shearing forces, then during flow and deformation of the fluid all shear forces are zero. Real fluids have finite viscosity and so for deformation of the fluid we consider viscosity and shearing stresses. Non-viscous fluids have zero viscosity and may or may not be incompressible. Flow of a real fluid is called viscous flow.

Real fluids are subdivided into Newtonian and non-Newtonian fluids. Water, air and gasoline are Newtonian fluids. A graph relating shear stress τ and shear rate du/dy is a straight line passing through the origin and the gradient gives the viscosity μ. Hence, Newtonian fluids are fluids in which shear stress is directly proportional to rate of deformation.

All other fluids in which shear stress is not directly proportional to deformation rate like toothpaste and paint are called non-Newtonian fluids.

Pseudoelastic fluids are fluids whose viscosity decreases with increasing rate of shear but the material deforms as soon as a shearing stress is applied, examples are polystyrene in organic solvents and metallic soaps in gasoline.

Dilatant fluids are those in which viscosity increases with the rate of shear, examples are butter and starch suspensions.

Bingham fluids are fluids which behave like solids until a minimum yield stress is exceeded and have a linear relation between shear stress and rate of deformation, examples are oil paints and clay suspension.

Figure 2 below is a graph of the relationship between shear stress, Ꭲ, and shear rate du/dy for Newtonian and non-Newtonian fluids.

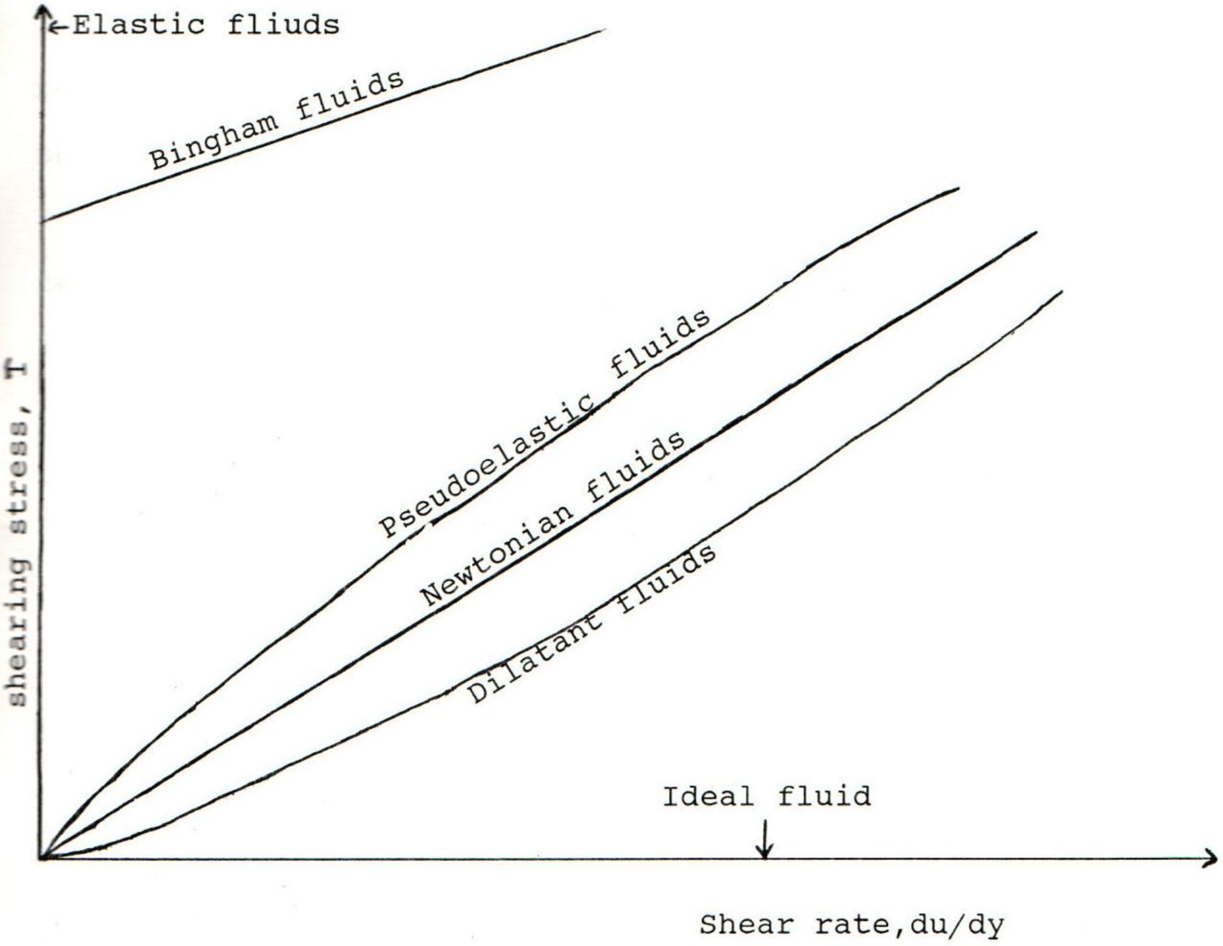


Fig 2: Flow curves for viscous behaviour in Newtonian and non-Newtonian fluids

1.4 Definition of fluid mechanics

Fluid mechanics deals with the behaviour of fluids at rest and in motion. The science of fluid mechanics is concerned with the motion of fluids and conditions affecting that motion.

Fluid kinematics is a branch of fluid mechanics dealing with the forces involved. It involves quantities like velocity, accerelation and rate of discharge. Fluid dynamics is a branch of fluid mechanics which deal with the forces acting on the fluids. It involves the application of Newton's second law of motion to the moving mass of fluid where forces like pressure, shear, gravity and inertia are involved.

Heat transfer and mass transfer occur in the fluid due to temperature and concentration difference.Rate of heat transfer is determined from the mechanics of fluid flow.

1.4.1 History of fluid mechanics before and in the 20th century

Flow of water in pipes and channels have been there for a long time. Hydraulics of both pipes and open channels for water supply was extensively used by the Romans although they were not aware of the laws associated with the resistance of pipes and channels to the flow of liquids.

Archimedes in 220 B.C used the bouyancy force to determine the gold content in the crown of king Hiero 11. In 1880 Reynolds introduced the concept of virtual turbulent stresses which was not sufficient to make theoretical analysis of turbulent flows. Osborne Reynolds came up with a series of papers describing the circumstances which determine whether the motion of water is direct or not and the law of resistance in parallel channels in 1883. Newton in the 17th century came up with his law of motion which was extensively used in the recognition of the nature of resistance of fluid flow and developed his own law of viscosity relating the shear force or stress to viscosity and relative motion of one layer of fluid adjacent to another layer. Venturi in the 18th and 19th centuries studied flow of fluids through conical reducing tubes and through expanding tubes for the purpose of reducing the turbulence and energy loses caused by the velocity changes in the flow. Navier-Stokes equations describing the motion of a viscous fluid were developed by Navier in 1827 and stokes independently in 1845. Mathematical difficulties involved in these equations could not allow theoretical treatment of viscous flows. Prandtl came up with the boundary layer theory which provided the link that had been missing between theory and practice in the present century. The science of fluid mechanics started to develop into two groups by the end of the 19th century. From Euler's equations of motion for a frictionless, non-viscous fluid,hydrodynamics came up which was of little practical importance. Hydraulics which to a large extent depended on experimental data differed greatly in its methods from hydrodynamics.

Prandtl in the present century tried to unite these two groups by coming up with the boundary layer theory in 1904. It was very important in the aerofoil theory and the science of gas dynamics together with the newly founded science of aerodynamics in 1935.

In 1921,Taylor brought the idea that the velocity of a fluid in turbulent motion was a random continuous function of position and time.Archimedes principle is of technical use in the design of displacement vessels,floatation gear and bathyscaphes. In 1935 Ludwig Prandtl introduced the mixing length theory which is of much use in turbulent flow problems.

In the 20th century, the field of research in fluid mechanics increased by experimental studies of boundary layer flow, heat transfer to and from flowing fluids and plasma multiphase flows.

1.4.2 Application of fluid mechanics

Fluid mechanics is not a subject to be studied only for academic interest, it is of much use both in our every day experiences and in all engineering and applied scientific studies. Flow of fluids in pipes and channels makes fluid mechanics of importance to civil engineers. Study of fluid machinery such as pumps, compressors, heat exchangers, jet and rocket engines make fluid mechanics of importance to mechanical engineers.

Aerodynamics is of much use to aeronautical and space engineers in the design of aircraft,missiles and rockets. Study of fluids in meteorology, hydrology and oceanography is important since the atmosphere and ocean are fluids. In modern engineering, fluid mechanics and electromagnetic theory are studied together as magnetohydrodynamics.

Design of transportation means requires application of the principles of fluid mechanics. Examples are aircraft for both subsonic and supersonic flight, ground effect machines, surface ships, submarines and automobiles.

Heating and ventilating systems for private homes, large office buildings and underground tunnels and the design of pipelines systems also require the knowledge of fluid mechanics.

Circulatory system of the body is a fluid system and so physiologists require basic principles of fluid mechanics in the design of artificial hearts, heart-lung machines and breathing aids.

1.5 The boundary layer.

1.5.1 The boundary layer concept:

Prandtl analyzed viscous flows by dividing the flow into two regions.One is a thin layer in the immediate neighbourhood of the solid wall where the influence of viscosity at high Reynolds number is confined called the boundary layer. The other region is that covering the rest of the flow region where the effect of viscosity is negligible and the fluid is treated as inviscid. In the boundary layer the velocity of the fluid increases from zero at the wall (no slip) to its full value. If we consider a uniform flow of compressible fluid flow with velocity U and U_∞ as its free stream velocity.

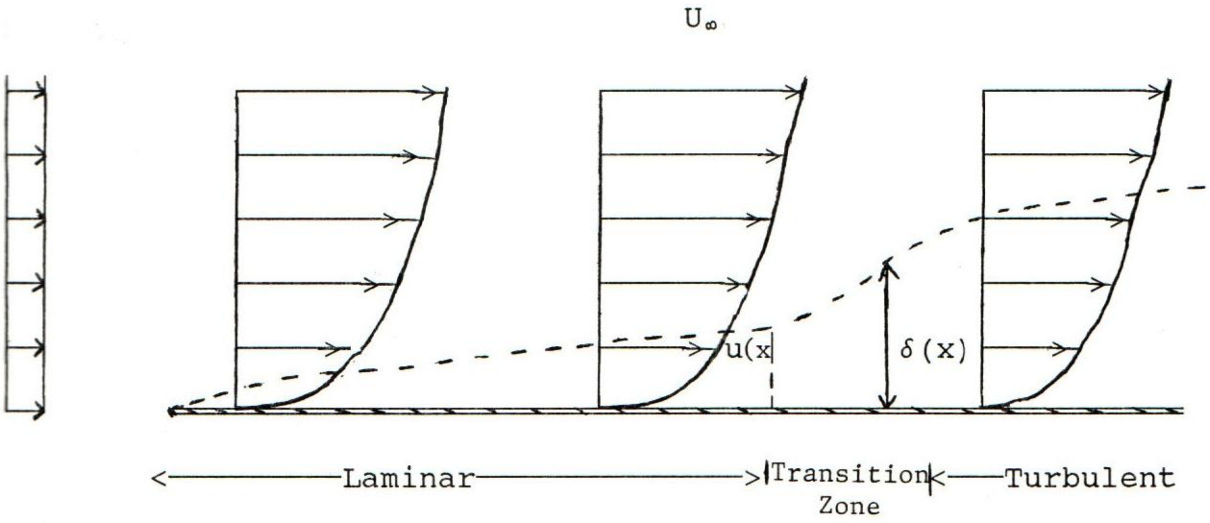


Fig 3: Sketch of boundary layer on a flat plate in parallel flow at zero incidence.

Even with very small viscosities (large Reynolds numbers) the frictional shearing stresses $\tau=\mu\ \delta u/\delta y$ in the boundary layer are considerable due to the large velocity gradient across the flow, outside the boundary layer they are very small. This shows that the field of flow of fluids with small viscosities can be divided into two regions. One is the thin boundary near the wall in which friction is considerable and the second is the region outside the boundary layer where friction forces can be neglected. This division simplifies the mathematical theory of the motion of fluids of low viscosity.

Flow in a boundary layer may be laminar or turbulent. There is no unique value of the Reynolds number at which transition from laminar to turbulent flow occurs in a boundary layer. Factors that affect the boundary layer transition are pressure gradient, surface roughness, heat transfer, body forces and freestream disturbances. As shown in figure 3 above, the boundary layer is laminar for a short distance downstream from the leading edge, transition occurs over a region of the plate rather than at a single line across the plate. The transition region extends downstream to the location where the boundary layer flow becomes completely turbulent. From the figure, we can see that the turbulent boundary layer grows at faster rate than the laminar layer.

1.5.2 Boundary layer thickness

The boundary layer thickness, $\delta(x)$, is the distance from the surface to the point where the velocity is within one percent of the free-stream velocity. In front of the leading edge of the plate the velocity distribution is uniform. With increasing distance from the leading edge in the downstream direction the thickness, $\delta(x)$ of the retarded layer increases continuously as increasing quantities of fluid become affected. The thickness of the boundary layer decreases with decreasing viscosity. In a few cases the boundary layer thickness increases considerably in the downstream direction and the flow in the boundary layer is reversed.

In summary ,the introduction of the boundary layer concept marked the beginning of the modern era of fluid mechanics. It is the foundation of our knowledge of the flow of fluids with small viscosities in technological fields especially in engineering. The boundary layer theory is of much use in aerodynamics by the study of its flow and the general effects on the flow around the body.

1.6 Some important dimensionless parameters.

The following are some independent dimensionless quantities which will be of much use in heat transfer equations:

(1) Reynolds number is the ratio of inertia to viscous force denoted by R_e

$$R_e = UD/\nu$$

where U is the characteristic velocity and D is the characteristic dimension.

(2) **Kinematic viscosity,** $\nu = \mu/\rho$

where μ is the viscosity and ρ is the density of the fluid

(3) **Grashof number** is a measure of the relative importance of bouyancy forces and viscous forces

$$G_r = g\beta L^3 \ (\blacktriangle T)/\nu^2$$

=(bouyancy forces) (inertia forces)/(viscous forces)

where $(\blacktriangle T)$ is the temperature difference

(4) **Thermal diffusivity** ;α

$$\alpha = k \ /\rho C_{p*}$$

where C_{p*} is the specific heat capacity at constant pressure

(5) **Prandtl number** is the ratio of kinematic viscosity to thermal diffusivity

$$P_r = \nu/\alpha = \mu C_{p*}/k = \nu/k/\rho C_{p*}$$

It is an important measure of relative importance of viscosity and heat conduction. P_r depends only on the properties of the medium.

(6) **Eckert number**, E_c is the ratio of kinematic energy to thermal energy

$$E_c = U^2/C_{p*}(\blacktriangle T)$$

(7) **Nusselt number** ,N_u is the ratio of thermal resistance to the convective thermal resistance of the fluid defined as

$$N_u = hL/K$$

Where h is the convective heat transfer coefficient and L is the representative length.

CHAPTER TWO

2.0 LITERATURE REVIEW

The following is an account of several investigations of heat transfer in natural convection which has been carried out in the past.

Schmidt and Beckmann [1930] wrote a theoretical report concerning the experimentally determined temperature and velocity field in natural convection on a vertical plate. Pohlhausen extended the problem by introducing the stream function putting $u^* = \delta\psi/\delta y^*$ and $v^* = -\delta\psi/\delta x^*$, then the partial differential equation for ψ can be reduced to an ordinary differential equation by a similarity transformation.

$$\eta = c\frac{y}{4\sqrt{x}} = 4\nu c x^{3/4} \S(\eta)$$

$$\text{where} \quad c = \left[g\frac{(T_w^* - T^*_\infty)}{4\nu^2 T^*_\infty} \right]^{\frac{1}{4}}$$

The temperature distribution is determined by the function T(Ç). Using the similarity transformation of various values of the prandtl number, a comparison between the calculated velocity and temperature distribution and those measured by Schmidt and Beckmann [1930] is seen to be in good agreement. Pohlhausen's calculations were extended by Schuh [1946] to the case of large Prandtl numbers like those ones in oils.

Sparrow and Gregg [1959] carried out investigations for the case of very small prandtl numbers and the limiting cases when $P_r \to 0$ and $P_r \to \infty$ were examined by Lefevre [1956], Sparrow and Gregg [1956], Vlient [1969] and Vlient and Liv [1956]. On their study of natural convection flow along a vertical heated porous plate, Saljinikov et al [1988] presented equations for laminar incompressible boundary layers with the case of generalized similarity principles, investigated the characteristic thickness and plotted potentials along the wall leading to heat flow, where the effects of temperature along the wall has also been considered.

Ishihera and Kaksuta [1989] did experiments on water at wide range of Reynolds and Prandtl numbers for laminar and combined convection between vertical parallel plates with uniform surface heat flux. They explained experimentally and numerically. Numerical solutions were presented for $P_r = 7.64$ (water) and $P_r = 0.005$ (liquid sodium) for various numerical combinations of G_r and R_e numbers.

Crane [1970] studied the fluid flow and heat transfer flow field due to a stretching boundary and found a closed form solution of Navier-stoke's equation. Crane also got a closed form solution for temperature distribution at $P_r = 1$.

Kato et al [1985] studied free convection heat transfer from a vertical plate of length L and thickness d and analyzed the problem numerically using finite difference method. Their equation closely approximates the results of average Nusselt number on a thin vertical plate.

The solution of the equation for the free convection boundary layer flow on a vertical flat plate with a prescribed power-law heating has been studied for small values of Prandtl number by Merkin [1989]. In his study, he showed that the boundary layer divides up into two regions. The plate temperature is determined by matching the two regions. A similar study was done by Rai et al [1991] who carried out a study on the free convection flow of an incompressible viscous fluid past an infinite vertical plate for impulsive as well as uniformly accelerated motion of the plate when the plate's temperature varies as the square root of the time. To obtain the expressions for the velocity and skin friction, the Laplace technique was applied. They also discussed the various parameters which have an influence in the problem on velocity field and skin friction. Eckert and Jackson [1951] performed a comparison between theoretical results on free convection with measurements on heated vertical cylinders and flat plates.

Yang [1960] discussed solutions in natural flows on the effect of suction or blowing on the rate of heat transfer from a vertical plate in natural convection. Temperature distributions on the surface of the plate of the form $T_w - T_\infty = T_1 x^n$ produced similar solutions with a differential equation whose solutions were found by Sparrow and Gregg [1958].

Reibe and King [1991] carried out studies on laminar natural convection heat transfer from inclined surfaces. Zeglimati et al [1991] similarly carried out an extensive investigation on the transient laminar free convection over an inclined wet flat plate and their results.

Chen et al [1986] performed an analysis of flow and heat transfer characteristics of laminar free convection in boundary layer flows from horizontal, inclined and vertical plates in which the wall temperature $T^*_w\ (x^*)$ varies as the power of the axial coordinates of the form

$$T^*_w\ (x^*) = T^*_\infty + a x^{*n}$$

The governing equations are converted into a dimensionless form by a non similar transformation and the resulting equations are solved by a finite difference method. Numerical results for fluids with Prandtl numbers of 7.00 and 0.7 have been presented for three respective exponent values under each of the non-uniform surface heating conditions.

Blasius [1908] obtained the solution for the laminar boundary layer on a flat plate. Blasius considered velocity profile u/U similar for all values of x plotted versus a non-dimensional distance from the wall by introducing the stream function ψ and a dimensionless streamfunction. He got a non-linear third order ordinary differential equation which was impossible to solve in closed form. Blasius solved this equation using a series expansion and Howarth [1938] solved this equation more precisely using numerical methods where numerical values were given in tabular form and velocity profile was obtained in dimensionless form by plotting u/U versus η whose profiles agree with the analytical solution.

Raptis and Kafousias [1982] presented an analysis of free convection and mass transfer flow through a porous medium bounded by an infinite vertical porous plate with free-stream velocity. In their study, the porous plate was subjected to a constant suction, the temperature and species concentration at the plate were constant and the flow was steady, which was of much use in geophysics.

Nogotov [1980] gave the basic equations for convective heat transfer in a theoretical manner together with the methods for the approximation of the equations and boundary conditions. He also gave an outline of the algorithms for solving the variation equations in steady and unsteady problems with a brief section on effects of compressibility with no numerical examples.

Yamatoto and Iwamura [1976] expressed the equations of flow through a highly porous medium. Raptis et al [1981] using the above equations studied the influences of the free convective flow and the mass transfer on the steady flow of viscous fluid through the porous medium which is bounded by a vertical and plane surface, when the temperature and the concentration on the surface are constant. Raptis et al [1982] also studied the influence of the free convective flow on the steady flow of the viscous fluid through the porous medium, when there is constant heat flux on the above mentioned surface. Recently Raptis [1983] studied the unsteady flow through a porous medium when the temperature of the surface from which the porous medium is bounded is not constant.

Elder [1965] did experiments on vertical layers which showed that boundary layer developed at each surface with fluid movement upwards at the heated surface and downward at the cooled surface.

Important contributions on free convection heat transfer have also been done by Weber [1977], Vafai and Tien [1985], Vafai et al [1980], Ostrach [1974] and Sharawi et al [1987].

3.0 CHAPTER THREE

3.1 Basic concepts of heat and heat transfer

Heat is the energy transferred from one body to another due to a temperature difference. The fundamental difference between heat and temperature is that heat is a form of energy and temperature is a measure of amount of energy which is present in a body. Heat transfer has been described as the study of the rates at which heat is exchanged between heat sources and receivers when treated independently.

3.2 Importance of heat transfer.

Transfer of energy in the form of heat is encountered in all phases of engineering work,industries and environment.

For mechanical engineers, the study of heat transfer is important because of its application in designs of internal combustion engines, refrigeration and air condition plants, heat exchangers, coolers, condensers, furnaces and preheaters.

Electrical engineers apply heat transfer in the design of cooling systems for motors, transformers and generators.

Civil engineers use heat transfer in the production of comfort cooling or comfort heating in the design of building structures like dams and tunnels.

Chemical engineers use heat transfer in freezing, boiling, evaporation and condensation processes. It is of much use in chemical process industry or petroleum refinery in the production of work from a combustible fuel or nuclear reaction which is converted into useful work by means of boilers, turbines, condensers, air heaters, water preheaters and pumps.

Solutions of the problems of thermal pollution created by the inevitable discharge of waste heat into the environment need the applications of the knowledge of heat transfer.

Thermal barrier in aerodynamics involves finding means of transferring away from the aircraft the enormous amounts of heat produced by the dissipative effect of the viscosity of the air.

3.3 Modes of heat transfer:

Heat always flow from a higher temperature to a lower temperature. There are three modes of heat transfer namely Conduction, convection and radiation.

(i) **Conduction:**

Conduction is transfer of heat within a body or between several bodies in physical contact in the absence of fluid motion. It always require material medium like solid ,liquid or gas and takes place without changing the position of its molecules. Then we can define conduction as heat transfer which is the result of molecular exchange of energy.

(ii) **Radiation:**

Radiation is the mode of transmission of heat between two bodies placed at a distance from each other without any transmitting medium. Heat transfer takes place due to radiation of electromagnetic waves which can pass through even the material substance.The distance between the two bodies is known as radiant medium or transmitter medium.Heat transfer by radiation between two surfaces separated by a non-absorbing medium depends on the temperature of the two bodies,their emmisivities and their relative positions.

(iii) **Convection:**

Convection is the process of transmission of heat from one part of a fluid to another by actual movement of the matter carrying heat energy from higher temperature region to the lower temperature region.

In convection,the motion of the fluid is caused by the differences in densities resulting from the difference in temperatures. Therefore, it can only take place in fluids. Hence convection is the process in which heat is transferred from solids to liquids or gases when radiation is not considered.

Types of convection.

(1) **Natural or free convection:**

It is the flow of a fluid over a hot or cold surface taking place due to difference in densities caused by the temperature difference. The force which acts on the fluid and causes it to flow is called the bouyancy force. This force depends upon the temperature difference of the molecules of the fluid. If the temperature difference is increased, the flow of heat increases.

This heat flow also depends upon the heat transfer coefficient. An example is the flow of heat from the condenser to surroundings in a domestic refrigerator.

(2) Forced convection:

It is the flow of a fluid over a hot or cold surface under external pressure. The force which acts on the fluid and causes it to flow is due to a fan or blower and is independent of the temperature difference. The heat transfer coefficient in this case is larger than that of natural convection. This type of heat transfer is used in all kinds of heat exchangers, furnaces with artificial draughts and refrigerators.

Summary

Many flows are subject to a combination of natural and forced convection. Heated jets or diffusion flames caused by blowing combustible gas from a vertical pipe are controlled by forced convection in the initial region and by bouyancy forces far from the jet or pipe exit.

Industrial smoke-stacks usually have a significant imposed momentum flux to assist the initial rise of the contaminant plume and the effluent from chemical and power plants usually enters a river with sufficient momentum to carry it away from the bank and towards the centre of the stream before bouyancy forces carry the contaminant to the surface or bottom of the river.

Bouyancy is of great importance in the environment where differences between land and air temperatures can give rise to complicated flow patterns and in enclosures such as ventilated and heated rooms or reactor configurations.

3.4 Material properties of importance in heat transfer.

3.4.1 Thermal conductivity

It is the heat conducted in unit time across unit area through unit thickness when a temperature difference of unity is maintained between opposite faces. It is a property of a material which varies with the temperature, that is

$$q/A \; \alpha \; \delta T^{*}/\delta n \quad \text{or}$$

$$q/\ A = k\delta T^{*}/\delta n$$

where K is the thermal conductivity and n is the direction under consideration.

3.4.2 Specific heat

It is the heat necessary to raise the temperature of a unit weight of a substance through 1^{0} c . Most heat transfer processes occur at constant pressure and so the specific heat at constant pressure C_{p*} is very important. Temperature dependence of the specific heat in gases is more pronounced than in solids and liquids.

3.4.3 Thermal diffusivity ;α

It characterises the rate of change in temperature in transient heat transfer process. The higher the rate of temperature propagation the higher the thermal diffusivity of a substance defined as

$$\alpha = K/\rho C_{p}.$$

3.4.4 Coefficient of thermal expansion ;ß

The driving force in free convection is that of gravity acting on fluid regions of differing density. These density differences exist because of differential thermal expansion resulting from temperature differences in the fluid field. The physical property of the fluid of significance in this process is the coefficient of thermal expansion.

$$\beta = 1/v\ (\delta v/\delta t)_p$$

$$= -1/\rho(\delta\rho/\delta t)_p$$

3.5 The thermal boundary layer:

If a solid body is placed in a fluid stream which is heated so that its temperature is above that of the surroundings, then the temperature of the stream will increase only over a thin layer in the neighbourhood of the body. This thin layer in the neighbourhood of the body is called the thermal boundary layer. If we consider flow over an isothermal flat plate.

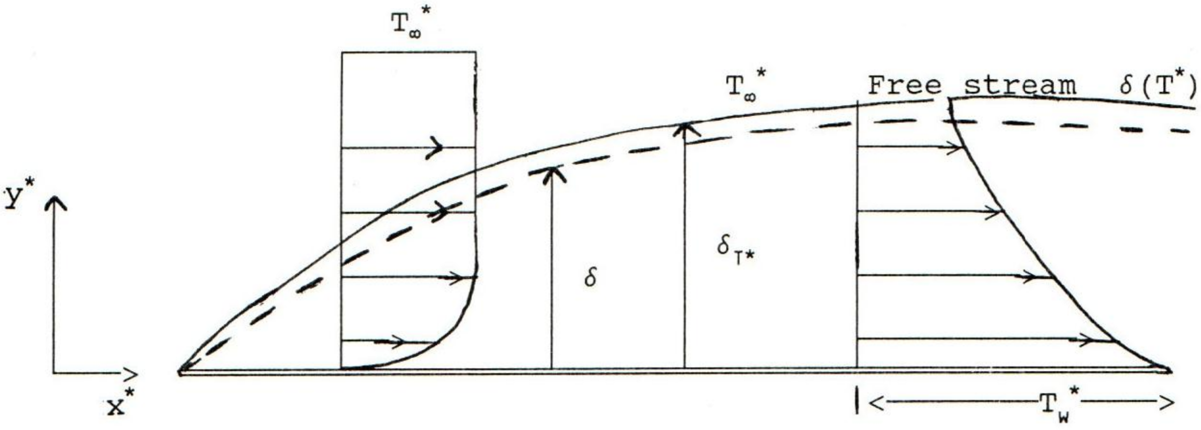


Fig 4: Thermal boundary layer on a flat horizontal plate

The fluid particles that come in contact with the plate achieve thermal equilibrium at the plate's surface temperature. These particles exchange energy with those in the adjoining fluid layer and a temperature gradient sets up in the fluid.The region of the fluid in which these temperature gradients exist is called the thermal boundary layer. The thickness of the boundary is denoted by δ_{T*} which is different from the velocity boundary layer thickness δ. As the distance from the leading edge increases, the heat transfer effects increases into the free stream and the thermal boundary layer becomes thicker.

3.6 The Navier-Stoke's equations

The three dimensional simplified Navier-stoke's equations for incompressible fluids where density and viscosity are constant are as follows:

$$\rho\left(\frac{\delta u^*}{\delta t^*}+u^*\frac{\delta u^*}{\delta x^*}+v^*\frac{\delta u^*}{\delta y^*}+w^*\frac{\delta u^*}{\delta z^*}\right)=f_{x^*}-\frac{\delta p^*}{\delta x^*}+\mu\left(\frac{\delta^2 u^*}{\delta x^{*2}}+\frac{\delta^2 u^*}{\delta y^{*2}}+\frac{\delta^2 u^*}{\delta z^{*2}}\right)\ldots\ldots\ldots \quad (3.1)$$

$$\rho\left(\frac{\delta v^*}{\delta t^*}+u^*\frac{\delta v^*}{\delta x^*}+v^*\frac{\delta v^*}{\delta y^*}+\frac{\delta v^*}{\delta z^*}\right)=f_{y^*}-\frac{\delta p^*}{\delta y^*}+\mu\left(\frac{\delta^2 v^*}{\delta x^{*2}}+\frac{\delta^2 v^*}{\delta y^{*2}}+\frac{\delta^2 v^*}{\delta z^{*2}}\right)\ldots\ldots \quad (3.2)$$

$$\rho\left(\frac{\delta w^*}{\delta t^*}+u^*\frac{\delta w^*}{\delta x^*}+v^*\frac{\delta w^*}{\delta y^*}+w^*\frac{\delta w^*}{\delta z^*}\right)=f_{z^*}-\frac{\delta p^*}{\delta z^*}+\mu\left(\frac{\delta^2 w^*}{\delta x^{*2}}+\frac{\delta^2 w^*}{\delta y^{*2}}+\frac{\delta^2 w^*}{\delta z^{*2}}\right)\ldots \quad (3.3)$$

The continuity equation

$$\frac{\delta u^*}{\delta x^*}+\frac{\delta v^*}{\delta y^*}+\frac{\delta w^*}{\delta z^*}=0 \qquad \ldots\ldots\ldots\ldots\ldots \quad (3.4)$$

The energy equation with constant thermal conductivity, k are:

$$\rho c_p^*\frac{DT^*}{Dt^*}=k\left[\frac{\delta^2 T^*}{\delta x^{*2}}+\frac{\delta^2 T^*}{\delta y^{*2}}+\frac{\delta^2 T^*}{\delta z^{*2}}\right]+\mu\,\Phi \quad \ldots\ldots \quad (3.5)$$

where

$$\Phi=2\left[\left(\frac{\delta u^*}{\delta x^*}\right)^2+\left(\frac{\delta v^*}{\delta y^*}\right)^2+\left(\frac{\delta w^*}{\delta z^*}\right)^2\right]+\left(\frac{\delta v^*}{\delta x^*}+\frac{\delta u^*}{\delta y^*}\right)^2+\left(\frac{\delta w^*}{\delta y^*}+\frac{\delta v^*}{\delta z^*}\right)^2$$

$$\left(\frac{\delta u^*}{\delta z^*}+\frac{\delta w^*}{\delta x^*}\right)^2-\frac{2}{3}\left(\frac{\delta u^*}{\delta x^*}+\frac{\delta v^*}{\delta y^*}+\frac{\delta w^*}{\delta z^*}\right)^2$$

where all the terms have their usual mathematical meanings [for derivation refer to schlichting **"Boundary layer theory"**, 1960].

3.7 Important Characteristics for temperature and velocity fields in free convection problems.

Experiments and analysis have shown that the thermal and viscous effects are concentrated in relatively thin layer adjacent to the solid surface and derivatives in the direction perpendicular to the solid surface are bigger than those in the flow directions. The density variation is taken into account only in the term that causes free convection currents and relatively small in other places and so can be neglected in the continuity equation.

In free convection, we assume that the flow is steady, low speed, density is negligible and the flow is of the thin boundary layer type. This help us to modify the Navier-stokes equations given above.

3.8 Thermal boundary layer equations

We concentrate on the free convection hydrodynamics and thermal boundary layer type flows in a vertical plate. The two dimensional continuity equation is given by

$$\frac{\delta u^*}{\delta x^*} + \frac{\delta v^*}{\delta y^*} = 0 \qquad \ldots\ldots\ldots\ldots (3.6)$$

momentum equation is

$$\rho(u^* \frac{\delta u^*}{\delta x^*} + v^* \frac{\delta u^*}{\delta y^*}) = f_{x*} - \frac{\delta p^*}{\delta x^*} + \mu (\frac{\delta^2 u^*}{\delta y^{*2}}) \ldots. (3.7)$$

where the right hand side of this equation represent the sum of all the forces on the control volume in the x^*- direction and f_{x*} is the body force per unit volume along the x^*- direction.

Energy equation for two dimensional steady flow is given by

$$\rho C_{p*} \left(u^* \frac{\delta T^*}{\delta x^*} + V^* \frac{\delta T^*}{\delta y^*} \right) = k \frac{\delta^2 T^*}{\delta y^{*2}} \quad \ldots\ldots\ldots \quad (3.8)$$

Considering a flow volume in the x^*-direction, the forces involved are shear forces, pressure forces and attraction of the mass contained in the control volume as shown below.

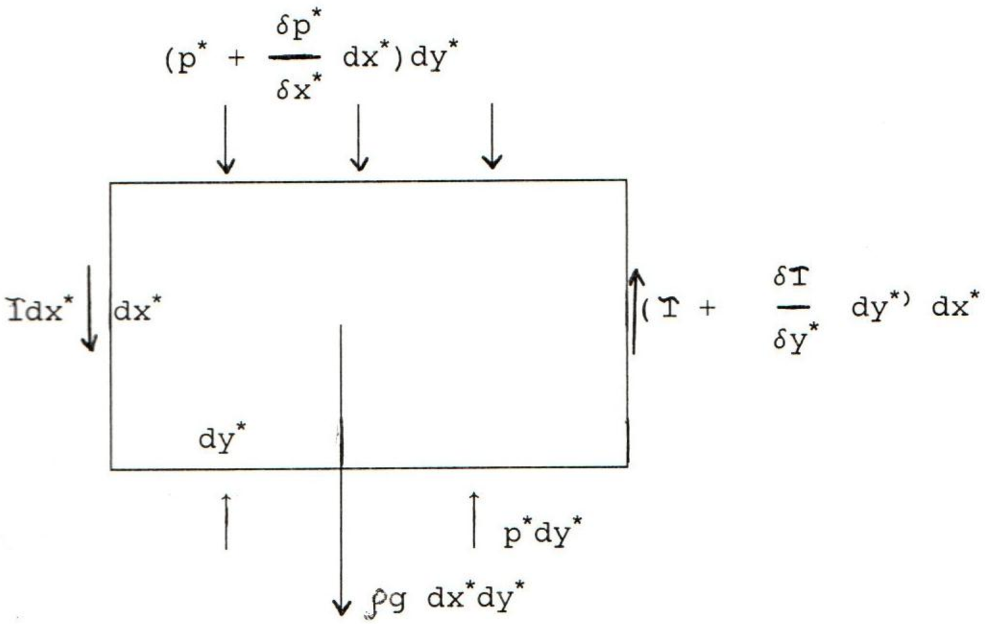


Fig 5:

Summing the forces in the x^*-direction, we have

$$\Sigma F_{x*} = p^* dy^* - (p^* + \frac{\delta p^*}{\delta x^*} dx^*) dy^* + (\tau + \frac{\delta \tau}{\delta y^*} dy^*) dx^*$$

$$- \rho g dx^* dy^* - \tau dx^*$$

$$= (- \frac{\delta p^*}{\delta x^*} + \frac{\delta \tau}{\delta y^*} - \rho g) dx^* dy^* \quad \ldots\ldots\ldots\ldots\ldots \quad (3.9)$$

The two-dimensional boundary layer flow for laminar flow is

$$\Tau = \mu \frac{\delta u^*}{\delta y^*}$$

substituting in equation (3.9), we get

$$\Sigma F_{x^*} = \left(- \frac{\delta p^*}{\delta x^*} + \frac{\delta \left(\mu \frac{\delta u^*}{\delta y^*} \right)}{\delta y^*} - \rho g \right) dx^* \, dy^*$$

$$= \left(- \frac{\delta p^*}{\delta x^*} + \mu \frac{\delta^2 u^*}{\delta y^{*2}} - \rho g \right) dx^* dy^* \quad \text{..........} \; (3.10)$$

Dividing equation (3.10) by volume elements, we get

$$\Sigma F_{x^*} = - \frac{\delta p^*}{\delta x^*} + \mu \frac{\delta^2 u^*}{\delta y^{*2}} - \rho g \quad \text{..........} \; (3.11)$$

Comparing equation (3.11) with equation (3.7), we get

$$f_{x^*} = - \rho g \quad \text{.........} \; (3.12)$$

which is the x^*-momentum equation balance for force.

For a point far outside

$$\frac{\delta^2 u^*}{\delta y^{*2}} = 0$$

For free convective process in the undisturbed fluid

$$\Sigma F_{x^*} = 0$$

with these two conditions, equation (3.11) reduces to

$$0 = - \left(\frac{\delta p^*}{\delta x^*} \right)_{\infty} - \rho_{\infty} g$$

Hence

$$-\left(\frac{\delta p^*}{\delta x^*}\right)_\infty = \rho_\infty g \quad \ldots\ldots\ldots\ldots \ (3.13)$$

Substituting (3.13), (3.12) into equation (3.7), we get

$$\rho\left[u^*\frac{\delta u^*}{\delta x^*} + v^*\frac{\delta u^*}{\delta y^*}\right] = \mu\left(\frac{\delta^2 u^*}{\delta y^{*2}}\right) + g(\rho_\infty - \rho)\ldots\ldots(3.14)$$

The last term represent the net upward force per unit volume. In the above equation we consider the net upward force by introducing a constant coefficient of thermal expansion, β

$$\beta = \left(\frac{1}{v}\frac{\delta v}{\delta T^*}\right)_{p^*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (3.15)$$

where

$V = \left(\frac{1}{\rho}\right)$ is the specific volume of the substance.

$$\left(\frac{\delta V}{\delta T^*}\right)_{p^*} = \frac{\delta}{\delta T^*}\left(\frac{1}{\rho}\right)_{p^*} = -\frac{1}{\rho^2}\left(\frac{\delta \rho}{\delta T^*}\right)_{p^*}$$

Equation (3.15) becomes

$$\beta = -\frac{1}{\rho}\left(\frac{\delta \rho}{\delta T^*}\right)_{p^*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (3.16)$$

Separating the variables in equation (3.16) and integrating treating β as a constant and assuming ρ in the denominator is extremely small, we get

$$\int_{T^*}^{T^*_\infty} \beta \, dT^* = -\int_{\rho}^{\rho_\infty} \frac{d\rho}{\rho}$$

$$\beta \, (T_\infty^* - T^*) = - \frac{(\rho_\infty - \rho)}{\rho}$$

that is

$$\beta \, g \rho (T_\infty^* - T^*) = - g \, (\rho_\infty - \rho) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.17)$$

substituting equation (3.17) into equation (3.14), we get

$$\rho \left[u^* \frac{\delta u^*}{\delta x^*} + v^* \frac{\delta u^*}{\delta y^*} \right] = \mu \frac{\delta^2 u^*}{\delta y^{*2}} + g \beta \rho \, (T^* - T_\infty^*) \ldots\ldots (\, 3.18)$$

The above equations lead to the conclusion that velocity and temperature fields in a free-convection two-dimensional flow depend on equations (3.6), (3.8) and (3.18). The differential equations for the velocity equation (3.14) and the thermal boundary layer equation (3.18) are similar in structure. The only difference are the last two terms in the equation of motion and in the last term in the temperature equation. In general temperature distribution depends on velocity distribution and the converse.

3.9 General properties of thermal boundary layer.

Temperature in natural convection depends on the flow field. When the heat due to friction and compression is neglected, the relation between the temperature field and velocity field depends on the prandtl number.

At high velocities friction and compression has to be taken into account and depends on the Eckert number.

Grashof number is important at very small velocities of flow, if the motion is caused by bouyancy forces the flow is independent of the Reynolds number.

Due to the interaction between the flow and the temperature field, the differential equation for calculating the velocity and temperature can be solved simultaneously. The pressure in the horizontal plate is equal to the gravitational pressure which is included in the Navier-stokes equations in the body force term and taken as constant.

3.9.1 Heat transfer by free convection

The flow pattern in free convection exists because of a temperature difference between the surface and the fluid. It is not caused by any other external energy. If the temperature of the solid is greater than that of the surrounding fluid, the fluid adjacent heats up leading to a decrease in the density and rising upward under the action of buoyancy forces. If the solid surface is colder than the surrounding fluid, the adjacent fluid is cooled and density increases resulting in a downward motion.

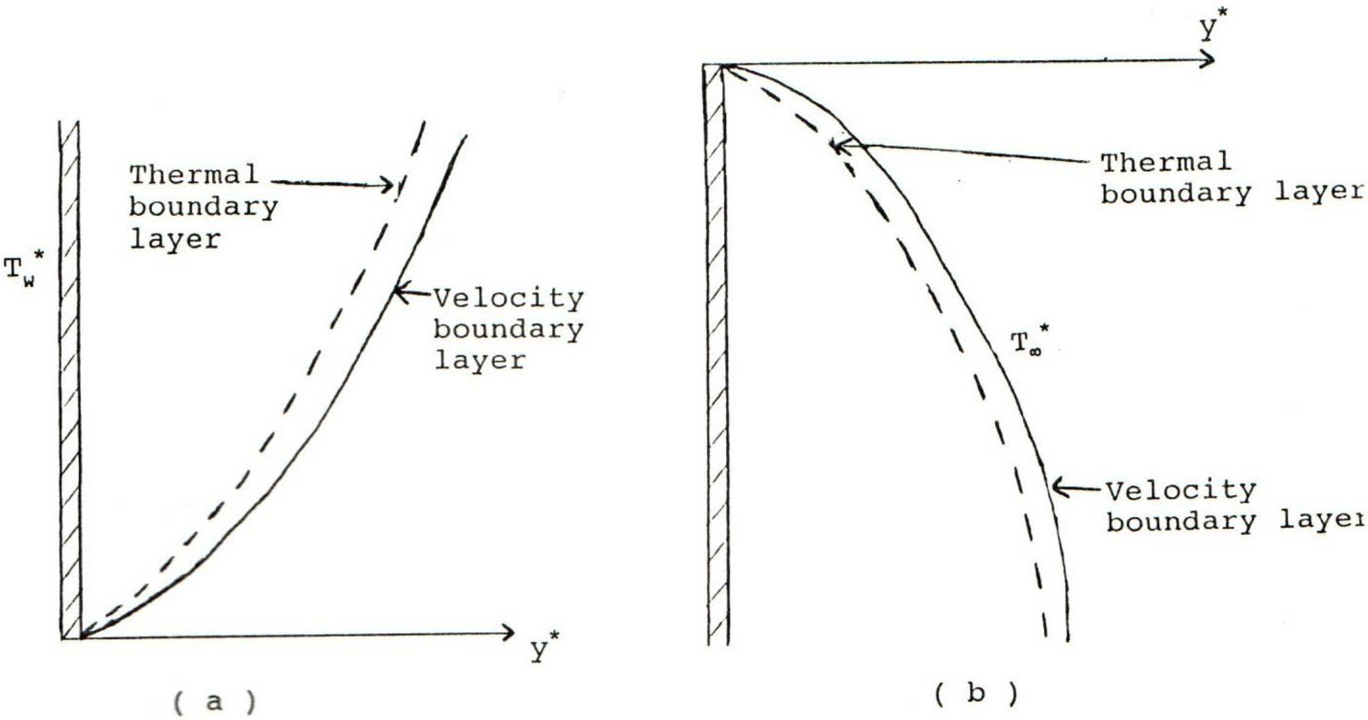


Fig 6: Coordinate system of free convection flow over a vertical surface (a) $T_w^* > T_\infty^*$ (b) $T_w^* < T_\infty^*$

For a vertical plate placed in a fluid at rest, if the surface temperature T_w^* of the plate is greater than the ambient temperature T_∞^*, the fluid adjacent to the surface is heated, becomes lighter and rises leading to the formation of velocity and temperature layers as shown in (a) above. If the wall temperature is less than the ambient temperature a similar boundary layer development occurs as shown in (b).

3.9.2 Application of free convection :

Heat ejection to the atmosphere, circulations in heated rooms, in the atmosphere and in bodies of water in the environment caused by buoyant forces is covered in natural convection.

Free convection can be observed in many thermal devices such as stoves, steam boilers and steam pipes.

Heat transfer from transmission lines, refrigeration coils and electric transformers.

Buoyancy is of great importance in energy and ecological applications.

3.9.3 Free convection heat transfer in a vertical plate.

If we consider a vertical flat plate of length, l whose temperature is T_w^* and surrounding temperature T_∞^*. When the plate is heated, some thermal and velocity boundary layer begin to form at the bottom. The pressure in each horizontal plane is equal to the gravitational pressure and is constant.

The only cause of motion is due to the difference between weight and bouyancy in the gravitational field of the earth.

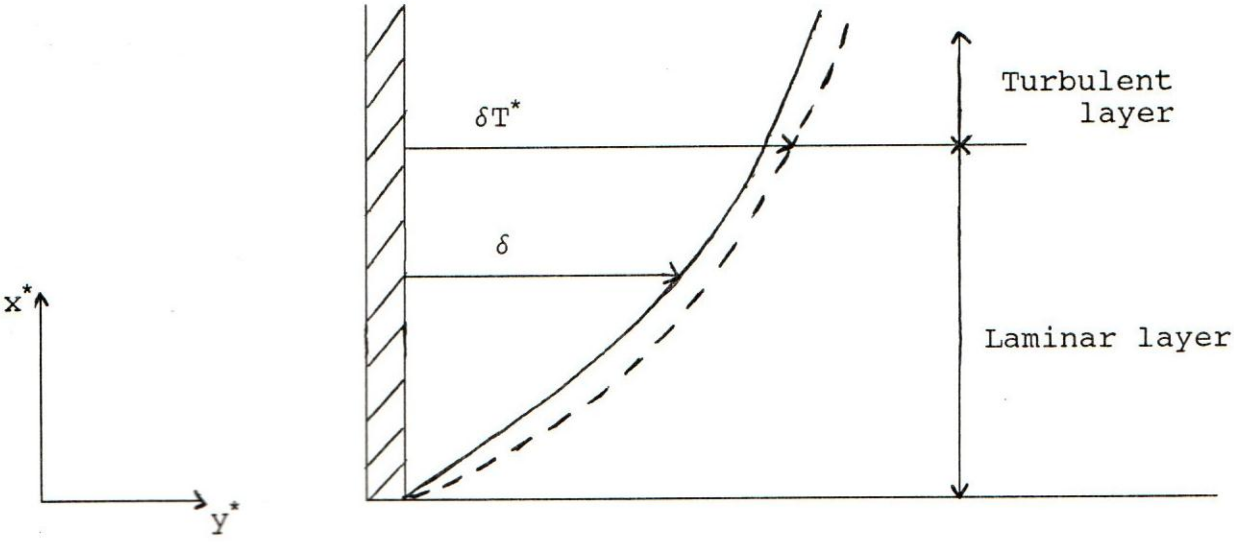


Fig 7:

The temperature and velocity profiles vary as shown below:

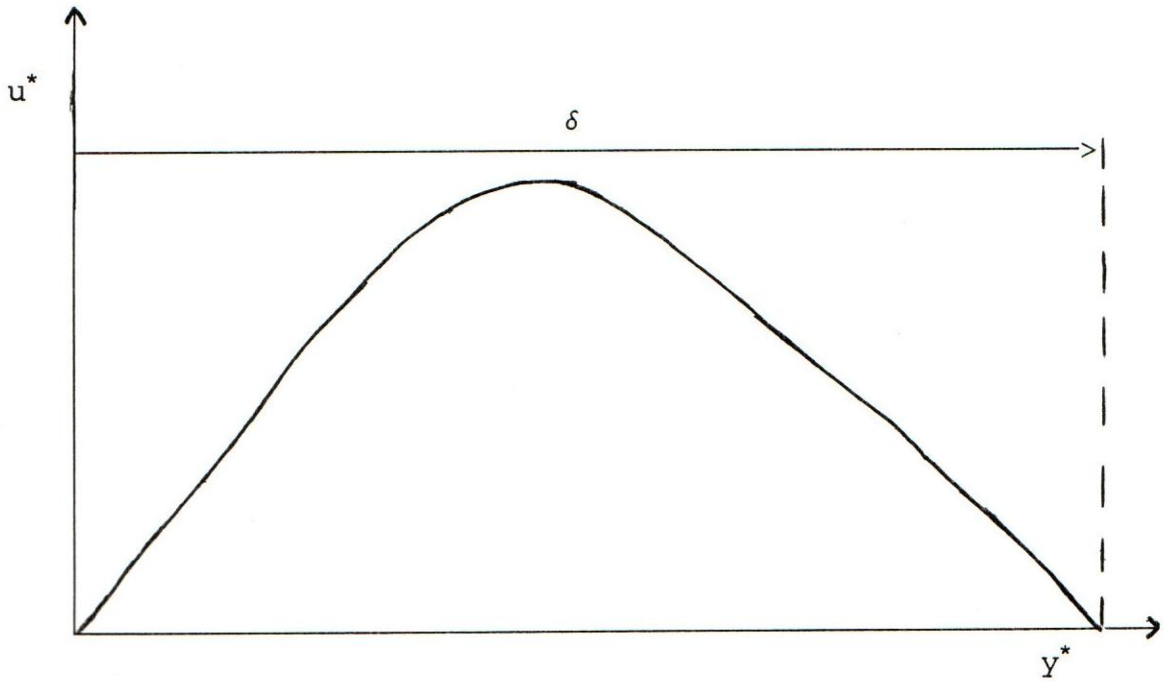


Fig 8a: Velocity profiles in natural convection heat transfer from a vertical plate

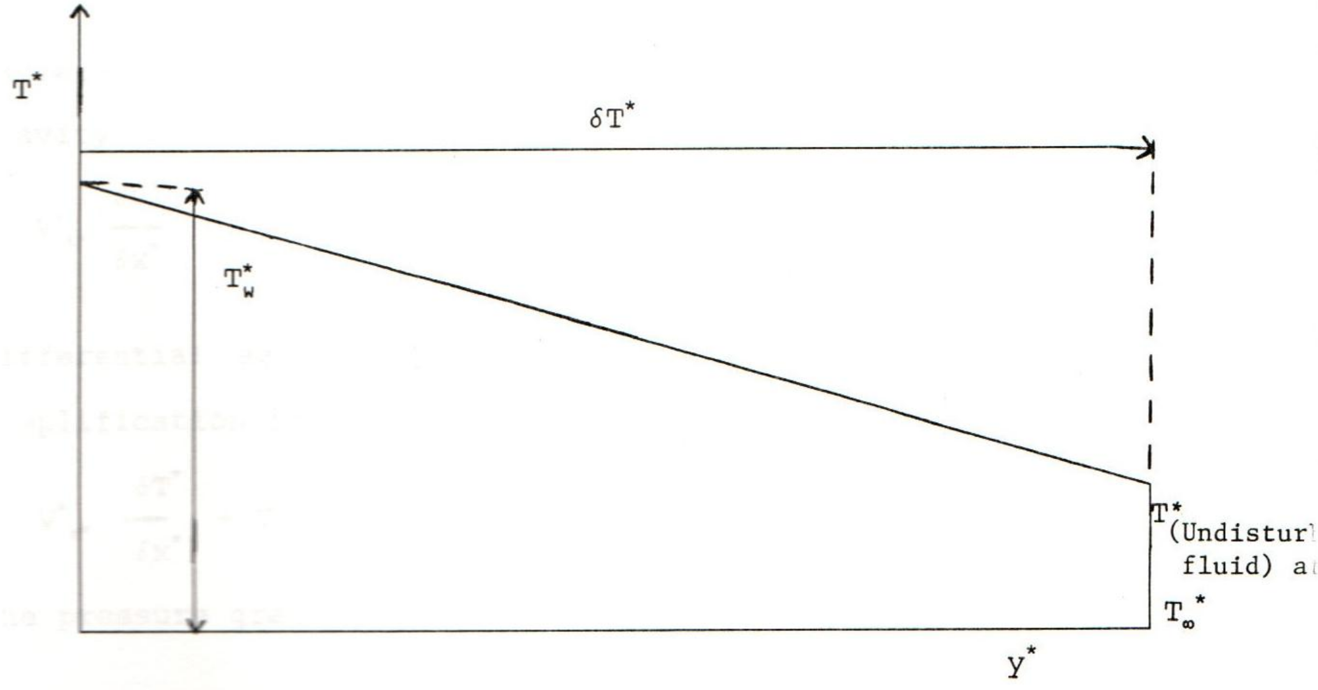


Fig 8b: Temperature profiles in natural convection heat transfer from a vertical plate.

Figure (8a) shows that the velocity is zero at the wall because of no slip condition. It increases to the maximum and decreases to zero due to undisturbed fluid. The original boundary layer is laminar but with increasing distance turbulence sets in and transition to turbulent boundary layer begins.Further up the plate,the boundary layer may become fully turbulent.

3.9.4 Exact solution of a flat plate at zero incidence.

To get the exact solution, we select the origin at the lower edge of the plate and the x^* and y^* directions parallel and normal to the plate, the boundary layer equations are:

Continuity equation

$$\frac{\delta V^*_{x^*}}{\delta x^*} + \frac{\delta V^*_{y^*}}{\delta y^*} = 0 \qquad \ldots\ldots\ldots\ldots (3.19)$$

Navier-stokes equation in x^* - direction with body forces due to gravity is

$$V^*_{x^*}\frac{\delta v^*_{x^*}}{\delta x^*} + V^*_{y^*}\frac{\delta V^*_{x^*}}{\delta y^*} = -\frac{1}{\rho}\frac{dp^*}{dx^*} + \nu\frac{\delta^2 V^*_{x^*}}{\delta y^{*2}} - g \qquad \ldots.(3.20)$$

Differential equation of heat convection with boundary layer simplification is

$$V^*_{x^*}\frac{\delta T^*}{\delta x^*} + V^*_{y^*}\frac{\delta T^*}{\delta y^*} = \alpha\frac{\delta^2 T^*}{\delta y^{*2}} \qquad \ldots\ldots\ldots(3.21)$$

The pressure gradient in the vertical direction is given by

$$\frac{dp^*}{dx^*} = -\rho_{\infty} g \qquad \ldots\ldots\ldots(3.22)$$

where ρ_∞ is the density of the surrounding fluid at T^*_∞. We modify equation (3.20) using (3.22) to obtain

$$V^*_{x*} \frac{\delta v^*_{x*}}{\delta x^*} + V_{y*} \frac{\delta V^*_{x*}}{\delta y^*} = \nu \frac{\delta^2 V^*_{x*}}{\delta y^{*2}} - \frac{g}{\rho} (\rho - \rho_\infty) \quad \ldots (3.23)$$

considering small differences the density term $(\rho - \rho_\infty)$ is related to the temperature difference $(T^* - T_\infty^*)$ through the volume expansion coefficient β by the relation

$$\beta \, (T^* - T_\infty^*) = - \frac{(\rho - \rho_\infty)}{\rho} \quad \ldots\ldots\ldots\ldots\ldots (3.24)$$

Equation (3.23) reduces to

$$V^*_{x*} \frac{\delta v^*_{x*}}{\delta x^*} + V^*_{y*} \frac{\delta V^*_{x*}}{\delta y^*} = \nu \frac{\delta^2 V^*_{x*}}{\delta y^{*2}} + g\beta \, (T^* - T_\infty^*) \quad \ldots . (3.25)$$

we solve equations (3.19), (3.21) and (3.25) subject to the boundary conditions.

When $y^* = 0$, $V^*_{x*} = V^*_{y*} = 0$, $T^* = T_w^*$

as $y^* \to \infty$, $V^*_{x*} \to V^*_{y*} \to 0$, $T^* \to T_\infty^*$

To simplify the problem, we introduce the stream function given as

$$V^*_{x*} = \frac{\delta \psi}{\delta y^*}, \qquad V^*_{y*} = - \frac{\delta \psi}{\delta x^*}$$

Equations (3.21) and (3.25) reduce to

$$\frac{\delta \psi}{\delta y^*} . \frac{\delta T^*}{\delta x^*} - \frac{\delta \psi}{\delta x^*} . \frac{\delta T^*}{\delta y^*} = \alpha \frac{\delta^2 T^*}{\delta y^{*2}} \quad \ldots\ldots\ldots\ldots (3.26)$$

$$\frac{\delta\psi}{\delta y^*} \cdot \frac{\delta^2\psi}{\delta x^*\delta y^*} - \frac{\delta\psi}{\delta x^*} \cdot \frac{\delta^2\psi}{\delta y^{*2}} = \nu \frac{\delta^3\psi}{\delta y^{*3}} + g\beta(T^*-T^*_\infty) \ldots \quad (3.27)$$

and define the dimensionless similarity variable

$$\eta = \frac{G\ y^*}{(\ x^*)^{1/4}}$$

where

$$G = \left[\frac{g\beta\ (T_w{}^*-T^*_\infty)}{4\ \nu^2}\right]^{\frac{1}{4}}$$

The functional dependence of the stream function on x^* and y^* axis is of the form:

$$4\ G\ (x^*)^{3/4}\ \nu\ f(\eta) \quad \ldots\ldots\ldots\ldots\ldots\ldots (\ 3.28\ )$$

and the dimensionless temperature parameter is defined as

$$T = \frac{T^*-T^*_\infty}{T_w{}^*-T^*_\infty} \quad \ldots\ldots\ldots\ldots\ldots (3.29)$$

which gives

$$V^*_{x^*} = \frac{\delta\psi}{\delta y^*} = 4\ \nu G^2\ x^{*1/2}\ \frac{df}{d\eta}$$

$$V^*_{y^*} = -\ \frac{\delta\psi}{\delta x^*} = \frac{\nu\ G}{x^{*1/4}}\left[\eta\ \frac{df}{d\eta} - 3f\right]$$

$$\frac{\delta V^*_{x^*}}{\delta x^*} = \frac{\delta^2\psi}{\delta x^*\delta y^*} = \frac{2\ \nu\ G^2}{x^{*1/2}}\left[\frac{df}{d\eta} - \frac{\eta}{2}\ \frac{d^2f}{d\eta^2}\right]$$

$$\frac{\delta V^*_{x^*}}{\delta y^*} = 4\,\nu\,G^3 x^{*1/4}\,\frac{d^2f}{dn^2} = \frac{\delta^2\psi}{\delta y^{*2}}$$

$$\frac{\delta^2 V^*_{x^*}}{\delta y^{*2}} = \frac{\delta^3\psi}{\delta y^{*3}} = 4\,\nu\,G^4\,\frac{d^3f}{d\eta^3}$$

$$\frac{\delta T^*}{\delta x^*} = -\,\frac{Gy^*\,(T_w^*-T^*_\infty)}{4x^{*5/4}}\,\frac{dT}{d\eta}$$

$$\frac{\delta T^*}{\delta y^*} = \frac{G(T_w^*-T^*_\infty)}{x^{*\,1/4}}\,\frac{dT}{d\eta}$$

$$\frac{\delta^2 T^*}{\delta y^{*2}} = \frac{G^2(T_w^*-T^*_\infty)}{x^{*\,1/2}}\,\frac{d^2T}{d\eta^2}$$

Substituting these values into equation (3.26) and (3.27) we obtain

$$\frac{d^3f}{d\eta^3} + 3f\,\frac{d^2f}{d\eta^2} - 2\left(\frac{df}{d\eta}\right)^2 + T = 0 \quad \ldots\ldots\ldots\ldots \quad (3.30)$$

$$\frac{d^2T}{d\eta^2} + 3fP_r\,\frac{dT}{d\eta} = 0 \quad \ldots\ldots\ldots\ldots \quad (3.31)$$

that is,

$$f^{111} + 3ff^{11} - 2\,(f^1)^2 + T = 0$$

$$T^{11} + 3fp_rT^1 = 0$$

where primes denote differentiation with respect to η. Thus the partial differential equations (3.26) and (3.27) are transformed into ordinary differential equations (3.30) and (3.31) with the following boundary conditions:

When $\eta = 0$, $\frac{df}{d\eta} = 0$ T=1, f= 0

as $\eta \to \infty$, $\frac{df}{d\eta} \longrightarrow 0$, f->0 , T->0.

Ostrach (1953) managed to solve equations (3.30) and (3.31) numerically using computer. The functions $f(\eta)$ and $T(\eta)$ have been obtained for Prandtl numbers ranging from 0.01 to 1000.

The Nusselt number

$$N_u = \frac{hx^*}{k} = \frac{K}{(T_w^* - T_\infty^*)} \left(\frac{\delta T^*}{\delta y^*} \right)_{y^*=0} \frac{x^*}{k} = Gx^{*3/4} \left(\frac{dT}{d\eta} \right)_{\eta=0}$$

Where Ostrach found the values of $\left(\frac{dT^*}{d\eta} \right)_{\eta=0}$ to fit the following empirical equation for prandtl numbers ranging from 0.01 to 1000

$$\left(\frac{dT}{d\eta} \right)_{\eta=0} = \frac{-(0.676\ P_r^{1/2})}{(0.861 + P_r)^{1/4}}$$

and so

$$N_u = \sqrt[4]{0.25}\ \frac{[g\beta(T_w^* - T_\infty^*)x^{*3}]^{1/4}}{\nu^2}\ \frac{0.76\ \sqrt{p_r}}{(0.861 + p_r)^{1/4}}$$

$$= \frac{\sqrt[4]{0.25}\ \sqrt[4]{G_r}\ (0.76)\ \sqrt{p_r}}{(0.861 + P_r)^{1/4}}$$

$$= \frac{0.48\ \sqrt[4]{G_r}\ \sqrt[2]{P_r}}{(0.861 + P_r)^{1/4}} \qquad .... (3.32)$$

Which is the exact solution for the above problem.

3.9.5 Approximate methods for a flat plate at zero incidence

Approximate methods are simpler in performing numerical calculations on thermal boundary layers than the above exact methods. The three equations for u^*, v^* and T^* in incompressible case ($\rho = \rho_\infty$ = constant) and for constant viscosity are:

Continuity equation

$$\frac{\delta u^*}{\delta x^*} + \frac{\delta v^*}{\delta y^*} = 0 \qquad \ldots\ldots\ldots\ldots\ (3.33)$$

Momentum equation

$$\rho_\infty\left(u^* \frac{\delta u^*}{\delta x^*} + v^* \frac{\delta u^*}{\delta y^*}\right) = \mu \frac{\delta^2 u^*}{\delta y^{*2}} - \frac{\delta p^*}{\delta x^*} - \rho_\infty g \beta (T^* - T_\infty^*) \ldots\ (3.34)$$

Energy equation

$$\rho_\infty\, c_p^*\left(u^* \frac{\delta T^*}{\delta x^*} + v^* \frac{\delta T^*}{\delta y^*}\right) = K \frac{\delta^2 T^*}{\delta y^{*2}} + \mu \left(\frac{\delta u^*}{\delta y^*}\right)^2 \ldots\ldots (3.35)$$

whose boundary conditions are:

when $y^* = 0$, $u^* = v^* = 0$, $T^* = T_w^*$ and $\frac{\delta T^*}{\delta y^*} = 0$

as $y^* \to \infty$: $u^* \to U_\infty^*$; $T^* \to T^*_\infty$

The velocity field is independent of the temperature field so that the two flow equations (3.33) and (3.34) can be solved first and the result used to evaluate the temperature field.

If we neglect the heat of friction $\mu\ (\delta u^*/\delta y^*)^2$ in equation (3.35) then a temperature field exists only if $T_w^* - T^*_\infty > 0$.

The properties of the fluid satisfy the equation

$$\alpha = \frac{k}{\rho c_{p*}} \quad \text{with } p_r = 1$$

This implies that for a flat plate at zero incidence parallel flow at small velocities, the temperature and velocity distributions are identical provided the prandtl number is unity that is, $(P_{r)} = 1$

$$\frac{T^* - T^*_w}{T_\infty^* - T_w^*} = \frac{u^*}{U^*_\infty} \quad \ldots\ldots\ldots\ldots\ldots (3.36)$$

We introduce the dimensionless coordinate $\eta \backsim y^* / \delta$ so that

$\eta = y^* \surd U^*_\infty / \nu x^*$ and the stream function

$\psi(x^*, y^*)$ such that, $\psi = \surd \nu x^* U^*_\infty f(\eta)$

where $f(\eta)$ denotes the dimensionless stream function

inserting

$$u^* = \frac{\delta \psi}{\delta y^*} = \frac{\delta \psi}{\delta \eta} \cdot \frac{\delta \eta}{\delta y^*} = U^*_\infty f^1 (\eta)$$

$$v^* = - \frac{\delta \psi}{\delta x^*} = \frac{1}{2} \frac{\surd \nu U^*_\infty}{x^*} (\eta f^1 - f)$$

in the equation $u^* \dfrac{\delta u^*}{\delta x^*} + v^* \dfrac{\delta u^*}{\delta y^*} = \nu \dfrac{\delta^2 u^*}{\delta y^{*2}}$

we get,

$$f f^{11} + 2 f^{111} = 0$$

when we include the effect of frictional heat in equation (3.35), the temperature distribution T^* is given by

$$\frac{d^2T^*}{d\eta^2} + \frac{p_r}{2} f \frac{dT^*}{d\eta} = -2P_r \frac{U^{*2}_\infty}{2C_{p*}} (f^{11})^2 \cdots (3.37)$$

with the boundary conditions

when $\eta = 0, \quad f = f^1 = 0$

as $\eta \to \infty, \quad f^1 \to 1$

The general solution of equation (3.37) by the superposition of two solutions is of the form

$$T - T^*_\infty = C\, T_1 + \frac{U^2_\infty}{2C_{p*}}\, T_2 \qquad \ldots\ldots\ldots\ldots\ldots\ (3.38)$$

where T, T_1 and T_2 are functions of n only and T_1 denotes the general solution of the homogeneous equation and T_2 denotes a particular solution of the non-homogeneous equation.

T_1 and T_2 satisfy the following equation

$$T_1^{11} + \tfrac{1}{2} P_r\, f T^1_1 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3.39)$$

and $T^{11}_2 + \tfrac{1}{2} P_r\, f\, T_2^1 = -\, 2P_r\, (f^{11})^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (3.40)$

with the boundary conditions:

$T_1 = 1 \quad$ at $\eta = 0$

$T_1 \to 0 \quad$ as $\eta \to \infty$

and

$T_2 = 0 \quad$ at $\eta = 0$

$T_2 \to 0 \quad$ as $\eta \to \infty$

From the value of $T_2\,(0)$ and $T = T_w^*$ for $\eta = 0$, we get

$$C = T_w^* - T^*_\infty - \frac{U_\infty^2}{2C_{p*}}\, T_2\,(0)$$

Equation (3.38) becomes

$$T - T^*_\infty = \left[(T_w^* - T_\infty^*) - (T_a - T_\infty^*) \right] T_1 + \frac{U_\infty^2}{2C_{p*}}\, T_2 \ldots \qquad (3.41)$$

where T_1 and T_2 are functions of η and P_r.

Then the dimensionless temperature distribution becomes:

$$\frac{T^* - T^*_\infty}{T_w^* - T_\infty^*} = \left[1 - \tfrac{1}{2}\, E_c\, b\, (P_r) \right] T_1 + \tfrac{1}{2}\, E_c\, T_2 \quad . \ (3.42)$$

where E_c = Eckert number = $\dfrac{U_\infty^{*2}}{C_{p*}(T_w^* - T_\infty^*)}$

and $b\ (P_r) = \sqrt{P_r} = 1$

Following Polhausen, we assume a polynomial of the fourth degree for the velocity function in terms of the dimensionless distance from the wall as:

$$\frac{u^*}{U} = f(\eta) = a\eta + b\eta^2 + d\eta^3 + e\eta^4 \ .... \ (3.43)$$

where $0 \le \eta \le 1$ and for $\eta > 1$, then $\dfrac{u^*}{U} = 1$

and distance from the wall $y^* = \delta(x^*)$.

To determine the four constants a, b, d and e we apply the following boundary conditions:

$$\text{For} \quad y^* = 0, \ \nu \frac{\delta^2 u^*}{\delta y^{*2}} = \frac{1}{\rho} \frac{dp^*}{dx^*} = - U \frac{dU^*}{dx^*}$$

$$\text{as} \quad y^* \to \infty, u^* = U; \quad \frac{\delta u^*}{\delta y^*} = 0, \quad \frac{\delta^2 u^*}{\delta y^{*2}} = 0 \qquad ... \ (3.44)$$

Introducing the dimensionless quantity $\S = \dfrac{\delta^2}{\nu} \dfrac{dU}{dx^*}$ .....(3.45)

to get

$$a = 2 + \frac{\S}{6}, \quad b = - \frac{\S}{2}, \quad d = -2 + \frac{\S}{2}, \quad e = 1 - \frac{\S}{6}$$

where § is the ratio of pressure forces to viscous forces.

The velocity profile becomes

$$\frac{u^*}{U} = F\ (\eta) + \S\ G\ (\eta)$$

$$= (2\eta - 2\eta^3 + \eta^4) + \frac{\S}{6} (\eta - 3\eta^2 + 3\eta^3 - \eta^4) \quad ....(3.46)$$

where

$$F(\eta) = 2\eta - 2\eta^3 + \eta^4$$

$$G(\eta) = - \frac{1}{6} [\eta - 3\eta^2 + 3\eta^3 - \eta^4] ....(3.47)$$

The two functions $F(\eta)$ and $G(\eta)$ given by equation (3.47) together compose the velocity distribution function. To get the temperature distribution, we integrate equation (3.35) from $y^* = 0$ to $y^* = \infty$, that is,

$$\frac{d}{dx^*} \int_0^\infty u^* (T^* - T^*_\infty) \ dy^* = - \alpha \left(\frac{\delta T^*}{\delta y^*} \right)_{y^*=0} \quad ......(3.48)$$

which is the heat flux equation where $\alpha = \dfrac{k}{\rho_\infty C_{p*}}$ is the thermal diffusivity.

To evaluate the integral on the left hand side of equation (3.48), we introduce the variables $\eta = y^*/\delta$ for the velocity boundary layer and $\eta_T = y^*/\delta_T$ for the thermal layer.

We denote their ratio by $\blacktriangle = \delta_T/\delta$.

From equation (3.42), we have

$$T-T_\infty^* = [T^*_W - T_\infty^*] \ [1 - 2\eta_T + 2\eta^3_T - \eta^4_T]$$

$$= (T_W^* - T_\infty^*) \ L \ (\eta_T) \quad ....(3.49)$$

From equation (3.46), we assume the velocity distribution to be of the form

$$u^* = U(x^*) \ [2\eta - 2\eta^3 + \eta^4] = U(x^*) \ F(\eta) \quad .... \ (3.50)$$

Equations (3.49) and (3.50) give the temperature and velocity expressions respectively by the approximate method.

Results

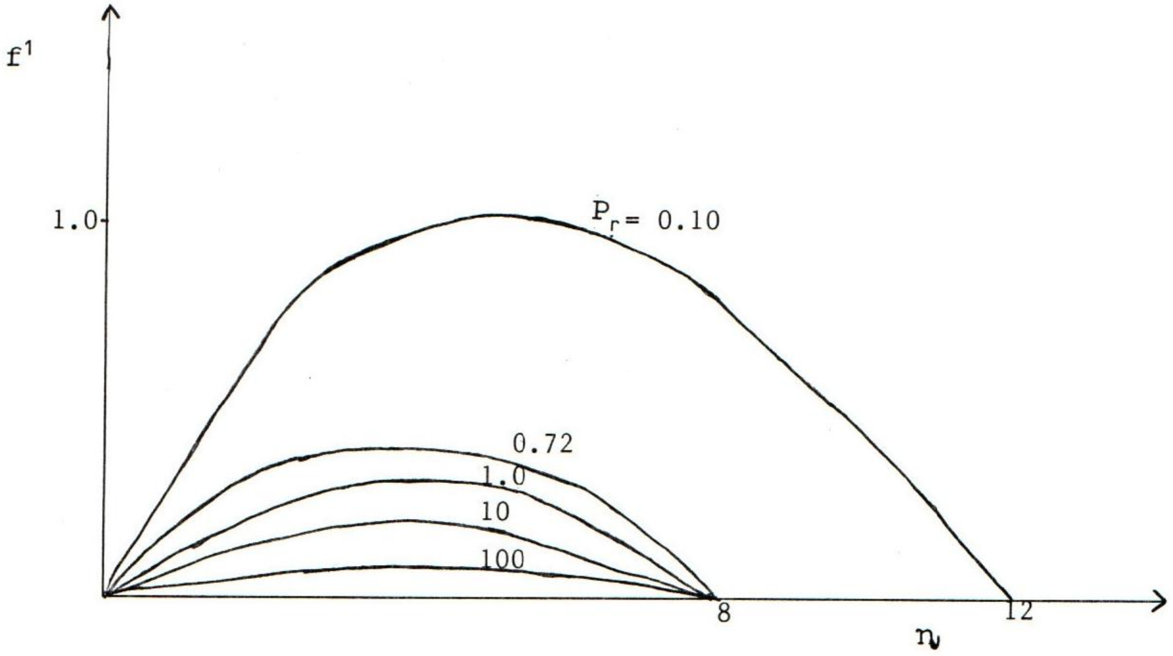


Fig 9: Velocity profiles for a natural convection boundary layer on a vertical plate with uniform wall temperature

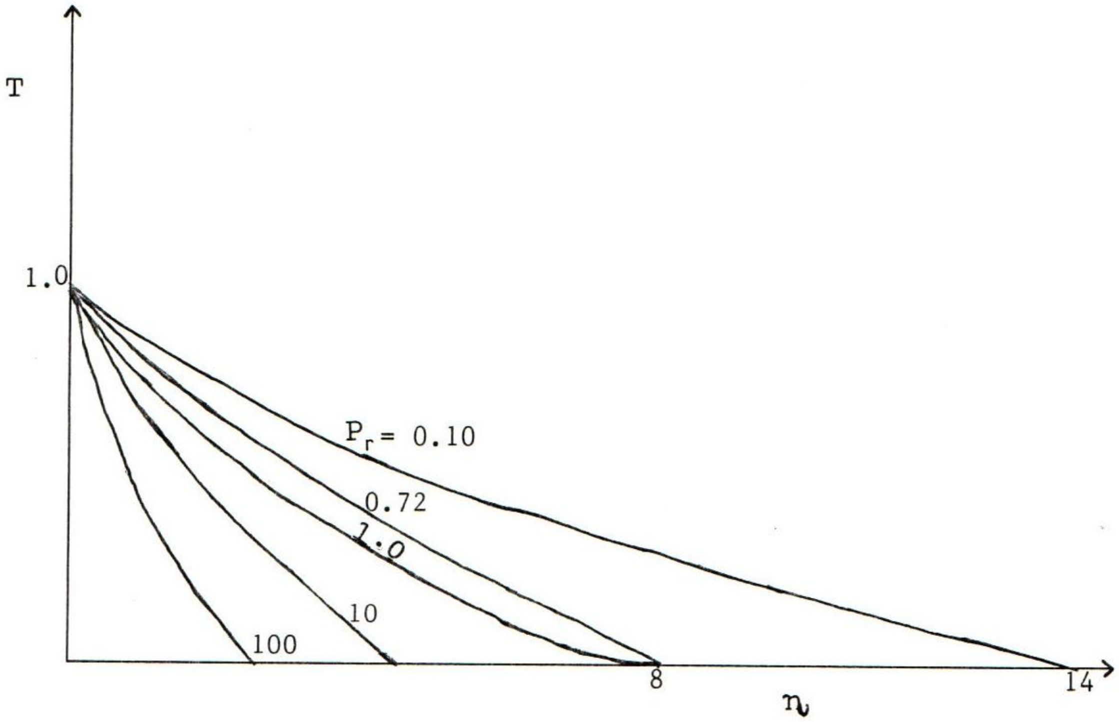


Fig 10: Temperature profiles for a free convection boundary layer on a vertical plate with uniform wall temperatures

CHAPTER FOUR

4.0 Free convective flow through porous medium

4.1 Introduction

The flow through a porous medium, under the influence of temperature differences is one of the most considerable and contemporary subjects, because it finds great applications in geothermy, geophysics and technology [1976, 1978].

Yamamoto and Iwamura [1976] expressed the equation of flow through a porous medium. Raptis et al [1981] using the above equations studied the influences of the free convective flow and the mass transfer on the steady flow of a viscous fluid through the porous medium, which is bounded by a vertical and plane surface, when the temperature and the concentration on the surface are constant. Raptis et al [1982] also studied the influence of the free convective flow on the steady flow of the viscous fluid through the porous medium, when there is constant heat flux on the above mentioned surface. Recently Raptis [1983] studied the unsteady flow through a porous medium, when the temperature of the surface from which the porous medium is bounded and non constant.

The purpose of present work is to study the effects of unsteady 2-dimensional free convective flow during the motion of a viscous incompressible fluid through a highly porous medium. The porous medium is bounded by a vertical surface of constant temperature. This surface absorbs the fluid with a constant velocity and the free stream velocity of the fluid vibrates above a mean constant value.

4.2 Mathematical analysis

Considering the unsteady 2-dimensional flow through a highly porous medium which is bounded by a vertical infinite plane surface. Assuming that the fluid is viscous and incompressible, the surface absorbs the fluid with a constant velocity and the velocity of the fluid far away from the surface vibrates about a mean value with direction parallel to the x^* - axis. All the fluid properties are assumed constant except that the influence of the density variation with temperature is considered only in the body force term. The x^* - axis is taken along the plane surface with direction opposite the direction of the gravity and the y^* - axis is taken to be normal to the surface.

The equations which govern the problem when the velocity and the temperature are functions of y^* and the time t^* are:

continuity equation

$$\frac{\delta v^*}{\delta y^*} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ (4.1)$$

momentum equation

$$\rho\left(\frac{\delta u^*}{\delta t^*} + v^* \frac{\delta u^*}{\delta y^*}\right) = -\frac{\delta p^*}{\delta x^*} - \rho g + \mu \frac{\delta^2 u^*}{\delta y^{*2}} - \frac{\mu}{k^*} u^* \ldots.\ (4.2)$$

Energy equation

$$\frac{\delta T^*}{\delta t^*} + v^* \frac{\delta T^*}{\delta y^*} = \frac{k}{\rho c_{p*}} \frac{\delta^2 T^*}{\delta y^{*2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.3)$$

Boundary conditions

for $y^* = 0$, $u^* = 0$, $v^* = -v_0 = \text{Constant}$, $T^* = T_w^*$

When $y^* \to \infty$, $u^* = U^* \to U\,(1 + \epsilon e^{iw^*t^*})$, $T^* \to T_\infty^*$ ....(4.4)

where u^* and v^* being the components of the velocity which are parallel to the x^* and y^* axes, respectively, ρ is the density of the fluid, p^* the pressure, g the acceleration due to gravity, μ the viscosity, $\mathbf{K}^*$ the permeability of the porous medium, T^* the temperature of the fluid, T_w^* the temperature of the surface, T_∞^* the temperature of the fluid far away from the surface, k the thermal conductivity of the fluid, c_{p*} the specific heat of the fluid at constant pressure, U a constant velocity, w^* the frequency of vibration of the fluid and $\epsilon(\epsilon<1)$ a constant quantity.

Equation (4.2), for the free stream, is reduced to

$$\rho \frac{dU^*}{dt^*} = -\frac{\delta p^*}{\delta x^*} - \rho_\infty g - \frac{\mu}{\mathbf{k}^*} U^* \qquad \ldots\ldots\ldots \quad (4.5)$$

On eliminating $\frac{\delta p^*}{\delta x^*}$ between (4.2) and (4.5) we have

$$\rho\left\{\frac{\delta u^*}{\delta t^*} + v^*\frac{\delta u^*}{\delta y^*}\right\} = \rho\frac{dU^*}{dt^*} + g\,(\rho_\infty - \rho) + \mu\frac{\delta^2 u^*}{\delta y^{*2}} + \frac{\mu^*}{k^*}\,(U^* - u^*)$$

This equation is reduced to

$$\rho\left(\frac{\delta u^*}{\delta t^*} + v^*\frac{\delta u^*}{\delta y^*}\right) = \rho\frac{dU^*}{dt^*} + g\beta\rho(T^* - T^*_\infty) + \mu\frac{\delta^2 u^*}{\delta y^{*2}} + \frac{\mu}{K^*}\,(U^* - u^*) \qquad \ldots\ldots\ldots\ldots\ (4.6)$$

by using the constitutive equation

$$\rho_\infty - \rho = \beta\rho\,(T^* - T^*_\infty)$$

where β is the volumetric coefficient of thermal expansion and ρ_∞ the density of the fluid far away the surface. Since the surface absorbs the fluid with a constant velocity, the continuity equation (4.1) gives

$$v^* = -\,v_0 = \text{constant}$$

Equation (4.6) then becomes

$$\frac{\delta u^*}{\delta t^*} - v_0\frac{\delta u^*}{\delta y^*} = \frac{dU^*}{dt^*} + g\beta(T^* - T^*_\infty) + \nu\frac{\delta^2 u^*}{\delta y^{*2}} + \frac{\nu}{K^*}\,(U^* - u^*) \qquad \ldots\ldots\ldots\ (4.7)$$

we introduce the non-dimensional quantities

$$u = \frac{u^*}{U_\infty}, \quad t = \frac{t^* v_0^2}{\nu}, \quad y = \frac{y^*\,v_0}{\nu}, \quad U = \frac{U^*}{U_\infty}$$

$$w = \frac{\nu w^*}{v_0^2}, \quad T = \frac{T^* - T^*_\infty}{T_w^* - T^*_\infty}$$

$$P_r = \frac{\rho\nu c_{p*}}{k} \qquad \text{(Prandtl number)}$$

$$G_r = \frac{\nu g\beta(T_w^* - T^*_\infty)}{U_\infty v^2_0} \qquad \text{(Grashof number)}$$

$$k = \frac{v_0^2 K^*}{\nu^2}$$

where ν is the kinematic viscosity.

With the help of the above non-dimensional variables, equations (4.7) and (4.3) are reduced to non-dimensional equations.

$$\frac{\delta u}{\delta t} - \frac{\delta u}{\delta y} = \frac{dU}{dt} + \frac{\delta^2 u}{\delta y^2} + \frac{1}{k}(U - u) + G_r T \quad \ldots\ldots\ldots \; (4.8)$$

$$p_r \left(\frac{\delta T}{\delta t} - \frac{\delta T}{\delta y}\right) = \frac{\delta^2 T}{\delta y^2} \quad \ldots\ldots\ldots \; (4.9)$$

by taking into account from (4.4) that $U^* = U(1 + \epsilon\, e^{iw^*t^*})$. The conditions (4.4) are reduced to:

For $y=0$, $u = 0$, $T = 1$

when $y \to \infty$, $U \to 1 + \epsilon e^{iwt}$, $T \to 0$ .... (4.10)

In order to solve the differential equations (4.8) and (4.9), we assume that

$$u(y,t) = u_0(y) + \epsilon e^{iwt}\, u_1(y) + \ldots\ldots\ldots\ldots\ldots \; (4.11)$$

$$T(y,t) = T_0(y) + \epsilon e^{iwt}\, T_1(y) + \ldots\ldots\ldots\ldots\ldots \; (4.12)$$

on subtituting equations (4.11) and (4.12) into equations (4.8) and (4.9) we get the following system of differential equations.

$$u_0^{11} + u_0^{1} - \frac{1}{k} u_0 = -\frac{1}{k} - G_r T_0 \quad \ldots\ldots\ldots\ldots \; (4.13)$$

$$u^{11}_1 + u_1^{1} - \left(\frac{1}{k} + iw\right) u_1 = -\left(\frac{1}{k} + iw\right) - G_r T_1 \ldots\ldots (4.14)$$

$$T_0^{11} + P_r T_0^{1} = 0 \quad \ldots\ldots\ldots \; (4.15)$$

$$T_1^{11} + P_r T_1^{1} - iwP_r T_1 = 0 \quad \ldots\ldots\ldots \; (4.16)$$

The conditions (4.10) are reduced to:

For $y = 0,\ u_0 = 0,\ u_1 = 0,\ T_0 = 1,\ T_1 = 0$

when $y \to \infty,\ u_0 \to 1,\ u_1 \to 1,\ T_0 \to 0,\ T_1 \to 0$ ....(4.17)

By solving the differential equations (4.13) - (4.16), under the conditions (4.17) and substituting the obtained solutions into (4.11) and (4.12) we have

$$u = \left[\frac{G_r}{(P_r + R_1)(P_r + R_2)} - 1\right] e^{R_1 y} - \frac{G_r}{(P_r + R_1)(P_r + R_2)} e^{-P_r y} + 1 +$$

$$\epsilon e^{iwt} \quad (1 - e^{R_3 y}) \qquad \text{.............} \ (4.18)$$

$$T = e^{-Pry} \qquad \text{.............} \ (4.19)$$

where $R_1 = \dfrac{-1 - \sqrt{1 + 4/k}}{2}$

$$R_2 = \frac{-1 + \sqrt{1 + 4/k}}{2}$$

$$R_3 = \frac{-1 - \sqrt{(1 + 4/k) + i4w}}{2}$$

It is also possible to write down the expression of the velocity (4.18) in the form:

$$u = u_0 + \epsilon e^{iwt} \quad (M_r + iM_i) \qquad \text{.............} \ (4.20)$$

where $M_r + iM_i = u_1$

when $wt = \pi/2$, the expression (4.20) for the velocity is given by

$$u = u_0 - \epsilon M_i \qquad \text{.............} \ (4.21)$$

Equation (4.18), in the form (4.21) takes the form

$$u = \left[\frac{G_r}{(P_r+R_1)(P_r+R_2)} - 1 \right] e^{R_1 y} - \frac{G_r}{(P_r+R_1)(P_r+R_2)} e^{-P_r y} + 1 + \epsilon e^{B1y} \operatorname{Sin}B2y$$

where

$$B_1 = \frac{-1 - \sqrt{\left[\frac{(1 + 4/\mathbf{K}) + \sqrt{(1 + 4/\mathbf{K})^2 + 16w^2}}{2} \right]}}{2}$$

$$B_2 = \frac{-\sqrt{\left[\frac{-(1 + 4/\mathbf{k}) + \sqrt{(1 + 4/\mathbf{k})^2 + 16w^2}}{2} \right]}}{2}$$

4.3 Conclusions

Figure II gives the variations for various values of the Grashof number (G_r) and the permeability parameter (**K**), when the Prandtl number (P_r) is equal to 0.71, which corresponds to the air.

From this figure we observe that the velocity increases when the Grashof number and the permeability parameter increases and the differences of the velocity are greater as the Grashof number increases.

$P_r=0.71, \quad \epsilon=0.2, \quad w=5$

	K	G_r
1	3	3
2	5	3
3	3	5
4	5	5

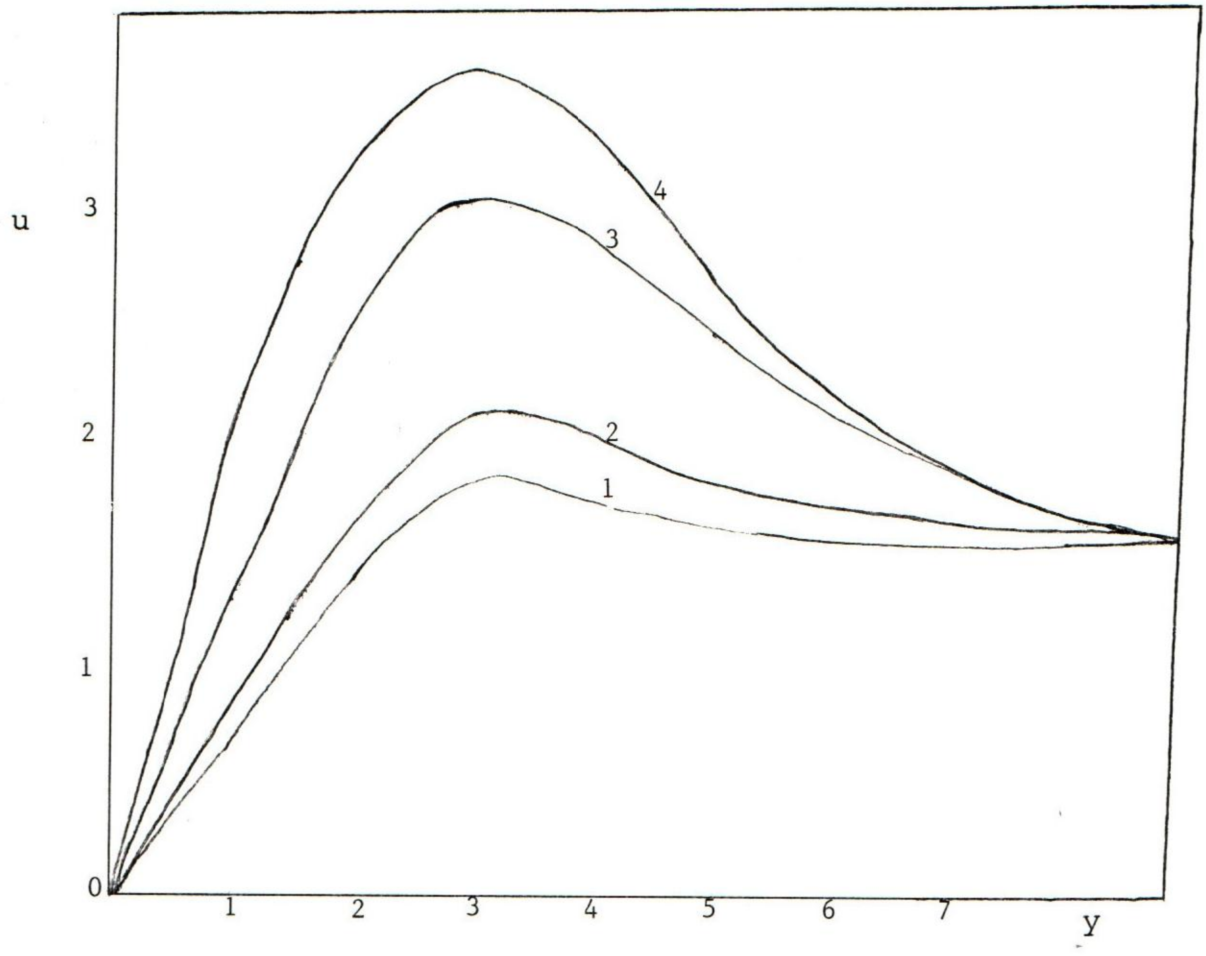


Fig 11: Velocity profile u

5.0 Nomenclature

C_{p^*}: Specific heat of the fluid at constant pressure

E_c : Eckert number

g : Gravitational attraction

G_r : Grashof number

k : Thermal conductivity

$\mathbf{K}, \mathbf{K}^*$: Permeability parameter

N_u : Nusselt number

P^* : Pressure

P_r : Prandtl number

T^* : Dimensional temperature of the fluid

T : Non-dimensional temperature

T_w^* : Temperature of the plate

T_∞^* : Temperature of the fluid in the free stream

▲T : Temperature difference

U : Constant velocity

U_∞^*: Free stream velocity

$V^*_{x^*}$: Velocity component in the x^* - direction

$V^*_{y^*}$: Velocity component in the y^* - direction

w^* : Frequency of vibration of the fluid

Greek symbols

ϵ : A constant quantity

§ : A constant quantity

δ : Heat source parameter

ß : Volumetric coefficient of thermal expansion

ρ : Density

ρ_∞: Density of the fluid far away from the surface

μ : Viscosity

ς : A constant quantity

α : Thermal diffusivity

ν : Kinematic viscosity

(x^*, y^*, z^*) : Coordinate system

(u^*, v^*, w^*) : Velocity component in x^*, y^*, z^* directions respectively

Subscripts

w : Vertical boundary wall

∞ : Distance far from the vertical boundary

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