MATHEMATICAL MODELS APPLICABLE IN BUSINESS INVESTMENTS

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DECLARATION:

This is my original work and has not been presented for a degree in any University.

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This work has been presented with my approval as supervisor

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ABSTRACT

The increasing reliance of finance and economics on mathematical models has made it necessary for managers, economists, and students to have some understanding of certain mathematical methods. However, the highly technical nature of many models often in applied in mathematics discourage managers and economists from using quantitative methods as tools in conceptualising and solving business problems. The field of investment is vast due to the fact that, theoretically, there is an infinitely wide choice with respect to a given investment project. The dimensions of choice are geared towards product/service, market, technology, equipment, and scale of production, time phasing, and location. The task is identifying investment opportunities, which are promising and which merit further examination and appraisal. There is need to come up with relations to solve discounted value of streams of income that the firm will generate in the future. This dissertation investigates the application of mathematical modelling in investment mathematics to help solve investment problems such as a series of cash flows. The dissertation uses systems of first order differential equations to relate investments and capital. It further uses linear difference equations in modifying the compound amount formula to obtain a simpler model for a series of cash flows. In addition, the annuities are solved by linear difference equations in areas of depositing and borrowing. The models discussed in dissertation have great importance in the sense that they consider a series of cash flows. The results obtained will help in management practices towards economic growth.

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NOMENCLATURE

CIF	Compound interest Factor
FV	Future Value
I_O	Initial Capital
I(t)	Invested Capital
K(o)	Capital at time t=0
K_*	Equilibrium Capital
K(t)	Capital at time t
K	A constant rate over which the parameter t is varied
m, n	Constant of proportionality
m	Periods in a year over which an amount is compounded
i	Interest rate
$PV^n_{a.I}$	Present deposited at the end of even periods
$PVDF^n_{a,I}$	Present value discount factor of an in year annuity immediate
R	Amount deposited at the end of even periods
r	Number of firms and also capitals

S_o	Initial amount at the account (principal)
S_t	Amount after time t
t	Time period expressed as a fraction or a multiple of a year

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CHAPTER ONE

INTRODUCTION

1.1 Background

The application of mathematics to financial and economic theory is widely used by managers and economists. For instance, in solving interest on a given capital and portfolio analysis for project appraisal. An investment decision-making involves a cash outlay with the aim of receiving, in return, future cash inflows. Decision about buying a new machine, building a factory or extending a warehouse are some cases of the investment decisions that may be made by industry. In order to help in making such decisions, and to ensure that they are consistent with each other, a common method of appraisal is required which can be applied equally to the whole spectrum of investment decisions and which should help to decide whether any particular investment will assist the company in maximising shareholder wealth.

If we take I= interest cost, i= interest rate, t = time (period) expressed as a fraction or a multiple of a year and p= principal, then I= itp. If the interest is compounded, then $FV^n = p(1+i)^n$ = p (CIF), where FV = future value, p = amount invested (principal), CIF= compound interest factor and n = period of investment . Suppose the interest is compounded a fraction of a year (say m year) then, $FV^n = p\left(1+\frac{i}{m}\right)^{mn}$. The model relating invested capital and increased capital is given by $I(t) = -K(o)\sqrt{m}\sin(\sqrt{mt})$, where I(t) = invested capital, K(o) = initial capital, m = constant of proportionality and t = time period.

If a person borrows some amount of money at a compound interest and wants to pay back in equal investment, say R then $S_t = \left(S_0 - \frac{R}{i}\right)(1+i)^t + \frac{R}{i}$, where S_t = the amount at end of t years, S_0 = borrowed amount, i = interest rate and R = equal investment. If the amount is paid back in n years, then S_n=0 and $R = S_0 \frac{i}{i-(1+i)^{-n}} = \frac{S_0}{a_{\frac{n}{i}}}$, where $a_{\frac{n}{i}}$ is the linearization factor which is the present value of an annuity of 1 per unit time for n-periods at interest. Suppose that an amount R is deposited at the end of even periods in a bank and S_t is the amount at the end of t periods. Then, $S_n = \frac{R\left[(1+i)^n - 1\right]}{i} = RS_{\frac{n}{i}}$, where $S_{\frac{n}{i}} = (1+i)^n a_{\frac{n}{i}}$. If S has to be paid back at the end of n- years, then it can be done by paying into a sinking fund an amount R per period, where $R = S.\frac{1}{S\frac{n}{i}}$ and $S_{\frac{n}{i}}$ is the sinking fund factor.

If S (t), Y (t) and I (t) are the savings, national income and investments respectively so that investment depends on the difference between the current income and that of the previous year then $I(t) = \beta\left[Y(t) - Y(t-1)\right]$, $\beta > 0$ where β is constant of proportionality. Suppose savings in the country depend on the national income then $S(t) = \alpha Y(t)$, $\alpha > 0$ is constant of proportionality. If all savings made are invested so that S(t) = I(t), then we have $Y(t) = \frac{\beta Y(t-1)}{\beta - \alpha}$, which has the solution $Y(t) = A\left(\frac{\beta}{\beta-\alpha}\right)^t = Y(0)\left(\frac{\beta}{\beta-\alpha}\right)^t$, where

$\beta > \alpha, \frac{\beta}{\beta-\alpha} > 1 and Y(t) \geq 0$.

The net present value of a project is given by

$NPV = \frac{CF_0}{(1+i)^0} + \frac{CF_1}{(1+i)^1} + --- + \frac{CF_n}{(1+i)^n} = \sum_{t=0}^{n} \frac{CF_t}{\{1+i\}^t}$ is depicted by Chandra (1987), where NPV = net present value, CF_t = cash flow occurring at the end of year t, n = life of the project, and i = cost of capital used as the discount rate. As depicted in Sarkis, *et.al* (1981), the net present value

$NPV = \left[\frac{CF_1}{(1+k_1)^1} + \frac{CF_2}{(1+k_2)^2} + --- + \frac{CF_n}{(1+k_n)^n}\right] - C_0$, where C_0 = cost of project (assumed to be incurred in its totality at time zero), k_t = cost of capital and

n = life of project

The internal rate of return is given by the expression

$C_0 - \left[\frac{CF_1}{(1+r^*)} + \frac{CF_2}{(1+r^*)} + --- + \frac{CF_n}{(1+r^*)}\right] = 0$, Where r* is the internal rate of return..

Or $\sum \frac{CF_t}{(1+r)^t} = 0$, where CF_t=cash flow at the end of year t, r= discount rate and n= life of the project. In the net present value calculation, we assume that the discount rate is known. In the internal rate of return calculation the net present value is set to be zero and determine the discount rate (internal rate of return), which satisfies the condition.

1.2 Statement of the problem

In Kenya, over 50 % of the resources are not properly invested. Economists and accountants have not come up with easier relationships to help solve investment problems. Further, relations have not been developed to solve a series of cash flows easily. The development of mathematical models for investment to facilitate faster computations, in terms of computer time, for investment returns is thus essential.

1.3 General Objectives

This dissertation simplifies presentation, summaries, clarifies concepts, and help integrated complex issues and relations of investment mathematics. We shall provide a perspective view on finance and economics by means of various mathematical models and techniques. The calculation of simple interest rate requires data on three variables: interest rate, expressed in percent on a yearly basis, or as a decimal; time period (a year , a fraction of a year , or a multiple thereof); principle or capital borrowed(lent).

1.4 Specific Objectives

a) To develop a mathematical model, which can be used to solve problems of investment for more than one capital invested in different areas (savings).

b) To derive mathematical models for capital invested in different portfolios when interest rates are different (portfolio analysis)

c) To develop mathematical models that would be used to analyse borrowing and lending investments especially when an equal sum of money is repaid at equal time intervals (annuity problem)

1.5 Definitions

Investment

Investment is the sacrifice of certain present value for (possibly uncertain) future value.

Annuities

Annuities are equal payments (or receipts) made (received) at equal time intervals during a period. There are two types of annuities: annuity- immediate where payments are made at the end of cash period for n periods; annuity - due where payments are made at the beginning of each period for n periods.

Interest

Interest is compensation for the temporary deprivation for the temporary deprivation of capital and for the risks associated with the change, real or perceived, in the value of that capital. In real terms, interest is the amount you receive on (pay for) capital that is lent (borrowed)

Interest rate

The interest rate is the percentage in interest paid (received) over a specified period, usually one year.

Savings

Saving is defined as foregone consumption.

Real versus financial investment

Some investments are simply transactions among people; others involve nature. The latter are "real" investment the former are not. Every investment can be conceived as an asset held by someone: the prospect of future returns. Some investments involve liabilities as well: someone else may have to provide the returns. A loan is a classic example; the lender has invested money but the investment is strictly financial in nature.

Models

A model is a mathematical function for depicting reality.

CHAPTER TWO

LITERATURE REVIEW

2.1 Background

A model may be considered as a simplified representation of certain aspects of a real system. A mathematical model is one created using mathematical concept such as functions and equations. When mathematical models are created, one moves from the real world into the abstract world of mathematical concepts, which is where the model is built. It must not be thought that for a particular problem there is one right and proper model. In fact, it is true and a remarkable demonstration of the power of mathematics, that the same abstract model can often be used for quite different physical situations. A mathematical model for finance or economics is an abstraction of the complexities of the real world designed to facilitate analysis of real problems, through a system of functional relationships among a set of variables.

Mathematical models can be classified into two categories: Computational models, based on mathematical structure yielding a general explicit solution or a range of solutions; and analytic models, which yield general optimality or equilibrium conditions. Both the categories of mathematical models are applied.

The development of a mathematical model requires a systematic approach such as.

- Problem identification and definition; hypothesis formulation.
- Statement of the assumptions.

- Identification of relevant variables based on understanding of the relevant environment and the nature of the problem with reference to the up-to-date literature on the subject under study.
- Development of the models to establish the mathematical relationships among the variables.
- Solution to the model.
- Testing the validity of the relations derived from the model.
- Conclusion based on the model.

Although the first four steps will be dealt with to some degree, beginning with the net present value (NPV) and the interval rate of return (IRR) models. The main emphasis is on the mathematical techniques for solving models.

2.2 Project Appraisal

There are many investment appraisal methods. We, however, consider the major ones, namely, the payback method; net present value method; and internal rate of return method.

2.2.1 Payback Method

The payback method yields the number of years it takes for the cash flow from a given investment to cover the cost of the investment. For instance, consider the criterion for project acceptance being a four-year (maximum) payback with initial out lay of $ 4,000. The payback systems is as follows:

TABLE 2.1: Payback systems for $ 4000:

Year	Cash flow
0	-$ 4000
1	$ 1000
2	$ 1000
3	$ 2000 payback period
4	$ 3000
5	$1000

We can see that it should be accepted because it pays back the initial outlay of $ 4000 within this period. However, in cases of more than one mutually exclusive project, then the project, which has faster speed of payback, is the preferred investment.

2.2.2 Working Capital

Most projects involve expenditure not only on capital equipment, but also on working capital. For instance, suppose a company was considering investing in a bread- making machine. Not only would the company have to incur the capital expenditure on the machine itself, but they would also have to invest in working capital: raw material stocks finished goods stocks and debtors. We should therefore recognise that, at the end of the projects life, the working capital should be recovered in full. This is because at the end of the project's life, the firm can run down its stocks of raw material and finished goods and, hopefully, all outstanding debtors will pay up. From this point of view, a question arises: should working capital be included, as part of the projects outlay and so be included in the payback calculation?

Let us take an example to answer the question. A Company is considering the purchase of a sausage machine. The machine would cost $ 12,000, and has an expected life of five years and a zero scap value (net of disposal costs) at the end of that time. In addition, an expenditure of $ 8,000 on working capital will be required throughout its life. The firm's management accountants have estimated the net, after –tax, operating cash flows of the project as follows:

TABLE 2.2: Payback systems of $ 12,000

Year	Operating cash flows
1	+ 6,000
2	+ 6,000
3	+ 6,000
4	+ 4,000
5	+3,000

The company evaluates investment opportunities using a three-year maximum payback criterion. The problem here is what set of cash flows should be evaluated using payback. To do this, we use two alternatives:

TABLE 2.3: Payback method with working capital

Year operating flow	Capital expenditure ($)	Working capital ($)	Operating cash flow ($)
0	-12,000	-8,000	
1			+ 6,000
2			+ 6,000
3			+ 6,000
4			+ 4000
5		+ 8,000	+3,000

TABLE 2.4: Payback methods without working capital

Year	Capital expenditure $	Operating cash flow $
0	-12,000	
1		+6,000
2		+6,000
3		+6,000
4		+4,000
5		+3,000

We can see that if the working capital is taken into account, the project should be rejected, as it takes more than three years to recover the total initial outlay of $ 20,000. However, if working capital is excluded from the analysis, the project is accepted as it has a two- year payback. The logic behind the approach of excluding working capital is that, payback is concerned about how long it will take the project to reach its break-even points where at least the outlay has been recovered. As a project's working capital is likely to be recovered whenever the project comes to the end of its life (that is, working capital is automatically paid back), it is excluded from the analysis of the break-even point.

The payback method has the following advantages to businessmen: it is very easy to calculate and understand; it is considered as a method for incorporating risk, that is, for investors who perceive investments with long payback periods as riskier than those with shorter payback periods; and the payback method is a measure of "liquidity" for the corporation. The shorter the payback period, the quicker the firm obtains its invested capital.

However, the payback method has the following limitations: it does not consider time value of money. The present value of cash flows from an investment is not the sum of the periodic cash flows; the payback method does not consider the cash flows that could be realised from an investment after its original cost has been recouped; and it does not offer a mechanism for the determination of the maximum acceptable payback period.

2.2.3 Net Present Value

The net present value of a project is equal to the sum of the present value of all the cash flows associated with the project. That is, it is the present value of "benefits" from an investment minus its cost. For equal cash flow and constant discount rate we have,

$$NPV = CF\left[\frac{1}{(1+i)^0}+\frac{1}{(1+i)^1}+\frac{1}{(1+i)^2}+.......\frac{1}{(1+i)^n}\right]-C_0$$

$$NPV = CF\sum_{t=0}^{n}\frac{1}{(1+i)^t}-C_0 \qquad (2.1)$$

where NPV = net present value

CF = cash flow

i = cost of capital used as the discount rate

n = life of the project

C_0 = cost of the project (assumed to be incurred in its totality at time zero)

To modify equation (2.1) we consider the present value factor $\sum_{t=0}^{n}\frac{1}{(1+i)^t}$, which can be written as $\sum_{t=0}^{n}(1+i)^{-t} = \left[\sum_{t=0}^{n}(1+i)^t\right]^{-1} = \left[\frac{e^{\frac{i}{k}}-1}{1-e^{-it/k}}\right]^{-1}$. Hence, we have

$$\sum_{t=0}^{n}\frac{1}{(1+i)^t} = \left[\frac{e^{\frac{i}{k}}-1}{1-e^{-it/k}}\right]^{-1} = \frac{1-e^{-it/k}}{e^{i/k}-1} \qquad (2.2)$$

Substituting equation (2.2) in (2.1) we obtain

$$NPV = CF\left(\frac{1-e^{\frac{-it}{k}}}{e^{\frac{i}{k}}-1}\right)-C_0 \qquad (2.3)$$

As an illustration, consider a ten-year investment having cash flow of \$ 5,000 annually and costing \$ 28,000 being considered by a certain management. The management cost of capital, I, is 10%. Determine the net present value of the project.

(a) Using the model (2.3) k = 1.050 from table 4.1, hence

$$NPV = 5{,}000\left(\frac{1-e^{\frac{-o.15\times 10}{1.05}}}{e^{\frac{0.10}{1.05}}-1}\right) - 28{,}000$$

$= 5{,}000(6.1467) - 28{,}000$

$= 30{,}733.30 - 28{,}000.00 = \$2{,}733.30$

(b) In comparison to the existing model (2.1)

$$NPV = 5{,}000\left(\sum_{t=1}^{10}\frac{1}{(1+0.10)^t}\right) - 28{,}000$$

$$= 5{,}000\left(\frac{1}{(1.1)^1}+\frac{1}{(1.1)^2}+.....+\frac{1}{(1.1)^{10}}\right) - 28{,}000$$

$= 5{,}000(0.9091 + 0.8264 + 0.7513 + 0.6830 + 0.6209 + 0.5645 + 0.5132 + 0.4665 + 0.4241$
$+ 0.3855) - 28{,}000$

$= 5{,}000(6.1446) - 28{,}000 = 30{,}723 - 28{,}000 = \$2{,}723$

The two results obtained give a close range of net present values, which shows that the model (2.3) is a shorter method to use.

The model summarised in equation (2.3) is based on the following assumptions: all quantities are measurable and known with certainty; i is the reinvestment rate; the project is independent of the other projects already under way in the firm: the cost of capital is independent of the size of the capital budget; and the cash flows are constant and independent the model is affected by controllable valuables such as: the cost of the project C_0; the cash flow from the project CF ; the cost of the capital; and the life of the project. The number **k** is the function of I and obtainable from table 4.1 these variables can be influenced by management decisions.

Decision Rules

The model allows the co-operate manager to make better capital budgeting decision such as :

i. If NPV < 0, the manager must reject the project

ii. If NPV > 0, the manager must accept the project

iii. If NPV = 0, the manager can either accept or reject the project

These decisions can have a measurable effect on the value of the firm. The higher the npv the greater is the value of the firm. When projects are ranked those with the highest npv are considered first. If the cash flows are not the same, that is, different cash flows then the model (2.3) becomes

$$NPV = CF_1\ell^{\frac{-i}{k}} + CF_2\ell^{\frac{-2i}{k}} + \ldots + CF_n\ell^{\frac{-in}{k}} - C_0 \qquad (2.4)$$

Graphical Interpretation of Net Present Value

In fig. 2.1, if management assess the company is total resources that is, liquidated value of the company at time in t, as OA and they have sort out all the investment alternatives available to them, expressed by the physical investment line AB, then using the npv methods of investment appraisal, they will invest in projects with positive or zero NPVS when discounted at the market rate of interest.

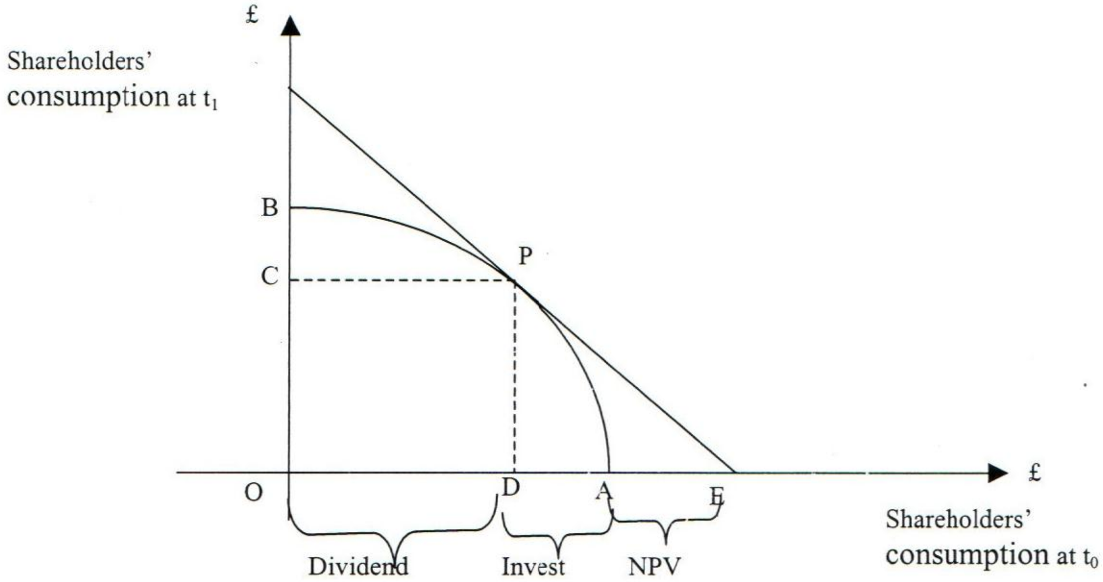


Fig. 2.1 Using the NPV rule

If the company does invest up to the optimal points p in fig 5.1, then the present value of the cash inflows generated by all the company's investment project undertakings is given by DE, the cash expenditure made on these investments is given by DA and so, by difference, AE represent the total net present value of the investment projects undertaken by the firm.

Consider fig.2.2 concerning leading and borrowing. Assuming that through the firm transfers wealth across time by lending and borrowing. The line CXD represents the opportunity for lending or borrowing. The slope of this line is (1+i), where i denote the year rate of interest. If the firm lends the present cash inflow of OA at an interest rate of percent, then the consumption in year 1 will be augmented by OA (1+i), which is BD. Alternatively, if the firm borrows against its future inflow, it can augment its current consumption by OB/(1+i) which is equal to AC.

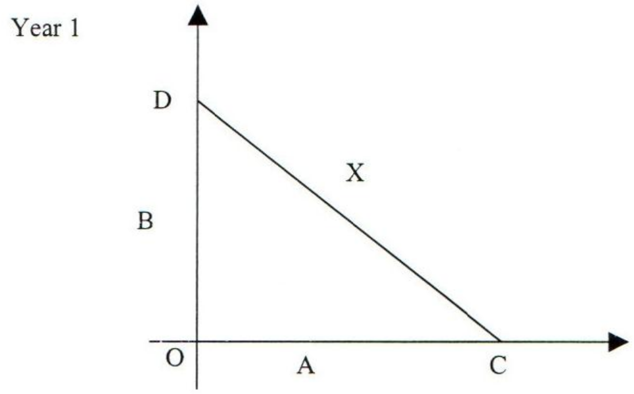


Fig.2.2 Lending -Borrowing opportunity.

In general, one can reach any points on the line CXD, by lending or borrowing. The net resent value criterion has the following considerable merits: it takes into accounts the time value of money; it considers the cash flow stream in its entirety;; it squares neatly the financial objective of maximisation of the wealth of stockholders. The net present value represents the contribution to the wealth of stockholders; and the net present value of various projects. In addition, the value of a package consisting of two projects, A, and B, will simply be the sum of the net present value of the individual project,

NPV (A+B) = NPV (A) + NPV (B)

Although the net present value has some weakness. The following are the limitations of the net present criterions. The ranking of projects on the net present value dimension is influenced by the discount rate; and it does not appear very meaningful to businessmen who are wanted to think in terms of rate of return measures.

2.2.4 Internal Rate of Return

Turning to the internal rate of return, we consider the net present value decisions rule. If a project has a positive net present value at a certain discount rate (says 10%) then the project return is actually greater than 10% whilst if the project has a negative net present value then its return is less than the discount rate, and if the project has a zero net present value then its return is equal to the discount rate.

The internal rate of return is the rate of discount which when applied to a projects cash flows produces a zero net present value. In other words, the internal rate of return is the rate, which equates the present value of cash flows from a project to the cost of the project. Considering equations (2.1) NPV = 0, hence we get

$$\sum_{t=0}^{n} \frac{CF_t}{(1+r)^t} - C_0 = 0$$

or

$$\frac{CF_1}{(1+r)^1} + \frac{CF_2}{(1+r)^2} + ... + \frac{CF_n}{(1+r)^n} = C_0 \qquad (2.5)$$

for constant (equal) cash flows say, CF, we get

$$CF\sum_{t=0}^{n} \frac{1}{(1+r)^t} = C_0 \qquad (2.6)$$

Since in equation (2.5) and (2.6), the present value factor is summed up, it is difficult to obtain the rate of return i. So a modified systems as below givens a simpler way of determining I, the rate internal return.

$$CF\left(\frac{1-\ell^{\frac{-rt}{k}}}{\ell^{\frac{r}{k}}-1}\right)=C_0 \qquad (2.7)$$

where r is the internal rate of return. The model assumes that: all quantities are measurable and known with certainty and r is both the discount rate and the reinvestment factor. The decision rule here is: if $r \geq i$, invest in project and if $r < i$, do not invest in project. The internal rate of return method is equivalent to the net present value method in every respect except for the discount factor. In the net present value calculation, the discount factor is known and is equal to the cost of capital. In the internal rate of return case, the discount factor is unknown and it may be much higher than the cost of capital. The assumption of the internal rate of return model that the firm will always be to reinvest at its internal rate of return is less realistic than the net present value model assumption of reinvestment at i.

An advantage of model (2.3) over the model (2.1) is that it considers investment proposals as a block, that is all the investment opportunities from a portfolio perspective. However, one difficulty arises from the fact that the net present value and internal rate of return methods assume that the firm is a profit maximises and their application ensures profit maximisation. Unconstrained profit maximisation however, is rarely possible. Firms are invariably constrained by cash availability, required earnings per share and environmental factors. These constraints can be accounted for in a net present value or internal rate of return maximisation model using linear programming, which will contribute to my further research.

2.3 Mathematical Models in Finance

Whereas household financial decisions are concerned with how to invest money, businesses typically need to raise money to finance their investments in real assets such as plant, equipment and technological know-how. Bodie *et.al*, (1989) have shown that there are two ways for businesses to raise money- either by borrowing it or taking in new partners. According to Sharpe (1981), investment is restricted to "real" investment of the sort that increases national output in the future. Adams *et.al (*2002); Ellipse, (2004) argue that investment mathematics has three parts namely, fundamental analysis of investment from a mathematical view point, relying heavily on compound interest techniques, provision of the necessary statistical background for specialists and applications of mathematical materials in modern portfolio theory and assets price indices, portfolio performance measurement, stochastic models and the theoretical pricing of options. Gitman, *et al.* (1988) say that time value of money refers to the fact that as long as an opportunity exists to earn interest, the value of money is affected by the point in time it is expected to be received. According to Sarkis *et.al.* (1981), calculation of interest cost requires data on three variables; interest rate; period and principal. Thus that, I = *it* P where I is interest cost, i is interest rate, t is time period and P is principal

Sarkis et.al. (1981) , Huntley et.al. (1989), Emery et.al. (1991) and Eduardo (1994) found out that the future value of an amount PV invested at I % compound every period for n years is $FV^n = PV(1+i)^n = PV(CIF)$, where $(1+i)^n = CIF$ is compound interest factor and PV is the present value. Sarkis *et.al.* (1981) have also shown that when the interest is compounded m times over a period of n years, then the future value will be $FV^n = PV\left(1+\frac{i}{m}\right)^{mn}$.

Annuity a pattern of equal returns (Sharpe, 1981) or payments. According to Mayo (1983), the Compound sum of annuity is

$$CS = I(1+I)^0 + I(1+I)^1 + I(1+I)^2 + \text{--------------} + I(1+I)^{n-1},$$

where I is annual payment. The present value of annuity is $PV = \sum_{t=1}^{n} \frac{I}{(1+i)^t}$. For different values of annual payment, then $PV = \frac{I_1}{(1+i)^1} + \frac{I_2}{(1+i)^2} + --- + \frac{I_n}{(1+i)^t} + \frac{P_n}{(1+i)^t}$

Sarkis *et.al.* (1981) have shown that the future value of an annuity immediate is $FV_{a,I}{}^n = \sum_{t=0}^{n-1} a(1+I)^t$ whose solution is $FV_{a,I}{}^n = a\left[\frac{(1+i)^n - 1}{i}\right]$, where **a** is the amount of the annuity. The present value of annuity immediate is given as

$$PV_{a,I}{}^n = a\left\{\frac{1 - \left[\frac{1}{(1+i)^n}\right]}{i}\right\}$$

Huntley et.al. (1982) argued that an investment opportunity is taken up only if the project value is substantially greater than zero, otherwise an alternative project is searched for with greater profit or less risk. Sarkis et.al. (1981), Lumby (1994) and Chandra (1987) have shown that the payback method is convenient for quick determination of period for recovery of initial cash outlay on the project. For payback criterion, the shorter the payback period, the more desirable the project. According to Chandra (1987), the net present value of a project is equal to sum of the present value of all cash flows associated with the project. That is,

$NPV = \sum_{t=0}^{n} \frac{CF_t}{(1+k)^t}$, where CF_t is cash flow occurring at the end of year t, NPV is net present value, n is life of the project and k is cost of capital used as the discount rate. Sarkis *et.al.* (1981), view the net present value as the present value of an investment minus its cost. That is, $NPV = \sum_{t=1}^{n} \frac{CF_t}{(1+k_t)^t} - C_0$, where C_0 is cost of project, NPV is net present value, k_t is cost of capital, CF is cash flow and n is life of project.

The internal rate of return is the value of **r** for which $\sum_{t=0}^{n} \frac{A}{(1+r)^t} = 0$, where **A** is cash flow and **r** is the internal rate of return, (Lumpy, 1994). Sarkis *et.al.* (1981) say that the internal rate of return is the rate which equates the present value of cash flows from a project to the cost of the project. That is, $C_0 - \left[\frac{CF_1}{(1+r^*)^1} + \frac{CF_2}{(1+r^*)^2} + --- + \frac{CF_n}{(1+r^*)^n} \right] = 0$, where r^* is the internal rate of return.

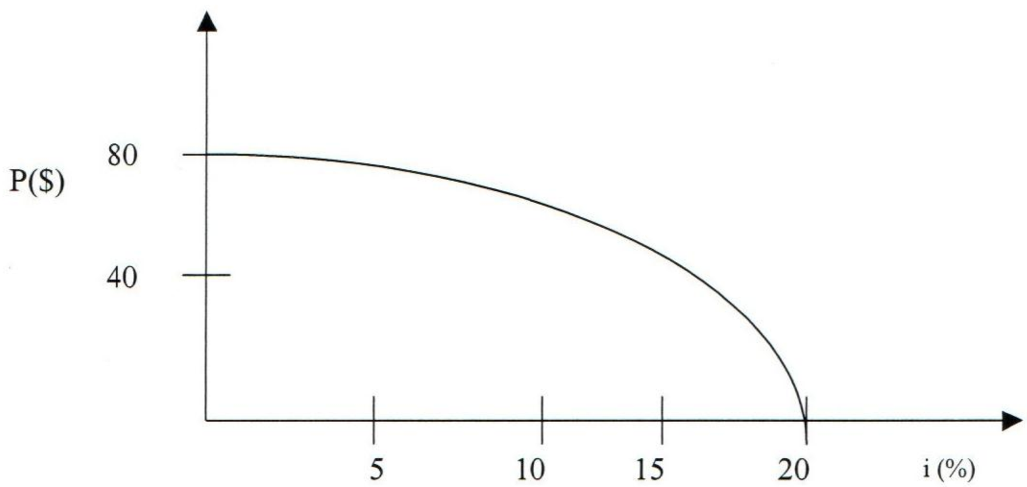
CHAPTER THREE

MODELS FOR BUSINESS INVESTIMENT:

3.1 Introduction

Very often, a firm has to decide between several alternative investment opportunities not all of which run for the same number of years. Accountants have found it easier to use the idea of the net present value rather than the value at the end of a project. Thus, a profit of \$36,000 in one year's time has a net present value of $\frac{\$36{,}000}{(1+i)}$, where i is the earning rate or interest rate. To see how important the value of the interest rate **i** is for the project value, fig 3.1 shows a graph of the present value **P** against **i** for fixed investment

Fig. 3.1: A graph of present value over interest rate



Probably the most convenient way of viewing this graph is to consider separately the cases where the net present value of the project is positive and negative. If P>0 you get greater return from going into the new venture than banking the original capital (whereP is the present value of the project). If P<0 then it would be financially advantageous to put the whole capital sum into the bank and pay off your workforce. In practice, an investment opportunity is taken up if the project value is substantially greater than zero, otherwise an alternative project with greater profit or less risk is searched for. `In business investments, the capital invested may either increase or decrease after some time. This is because of two basic dimensions of financial analysis, return, and risk. The cost is not considered because it is the inverse of return.

3.2 Variable Investment

Suppose that an investment leads to increased capital so that if **I (t)** is invested and **K(t)** is the capital at time **t**, then using the ordinary differential equations, the rate of change in capital is equal to the amount invested, that is

$$\frac{dK(t)}{dt} = I(t) \tag{3.1}$$

Also, if the rate of change in investment is proportional to the deviation from the equilibrium capital, then

$$\frac{dI(t)}{dt} = m\left(k(t) - k_*\right) \quad \text{or} \quad \frac{dI(t)}{dt} = -mK(t), \tag{3.2}$$

where K(t)= $k_* - k(t)$ and k_* is the equilibrium capital. The number 'm' is the appreciation factor for the capital employed.

By applying the product rule in equation (3.1), and then integrating, we obtain

$I^2 = m\left(K_0^2 - K^2\right)$, where $K_0 = K(o) and I_0 = 0$.

Thus, (3.1) can be written as

$$\frac{dk(t)}{dt} = -\sqrt{m\left(K_0^2 - K^2\right)} \tag{3.3}$$

which leads to $I(t) = -K(0)\sqrt{m}\sin(\sqrt{m})t$

If the investment is reduced by excess capital and not less capital, then this leads to

$$\frac{dk(t)}{dt} = I(t) \tag{3.4}$$

and

$$\frac{dI(t)}{dt} = -mk(t) - nI(t) \tag{3.5}$$

where `n' is a constant of proportionality for reduction in capital. Equations (3.4) and (3.5) reduce to;

$$I\frac{dI(t)}{dt} + mK(t) + nI(t) = 0 \text{ or } \frac{d^2K(t)}{dt^2} + n\frac{dK(t)}{dt} + mK(t) = 0$$

which is a second order ordinary differential equation governing investments.

In real terms 'n' represents the depreciation factor of the capital invested.

3.2.1 Alternative Model

Consider a case where capital is invested in two different businesses with the goal of knowing the amount that would be obtained from the investments after some time **t**. The first assumption is that the investments lead to increased capital in each case. Let $I_1(t)$ and $I_2(t)$ be the capital invested, and also $K_1(t)$ and $K_2(t)$ be the increased capital at time **t** respectively. We assume that the rate of change in capital is equal to the sum of the amount invested, so that (3.1) becomes

$$\frac{dK(t)}{dt} = I_1(t) + I_2(t)$$

where

$$K(t) = K_1(t) + K_2(t)$$

Under the assumption that the rate of change in investments is proportional to the deviation from the equilibrium capital, equation (3.2) becomes

$$\frac{d(I_1(t)+I_2(t))}{dt} = \frac{m}{2}(K(t) - K_*)$$

or

$$\frac{d(I_1(t)+I_2(t))}{dt} = -\frac{m}{2}K(t) \qquad (3.6)$$

where K(t)= $k_* - k(t)$ and K(t)= $K_1(t) + K_2(t)$

Equation (3.6) is a first order linear ordinary differential equation. Applying the product rule and then integrating (3.6) leads to

$$I_1^2 + I_2^2 = \frac{m}{2}(K_o^2 - K^2)$$

where

$$K_o = K(0)$$
$$I_o = 0$$

Now using (3.3),

$$\left.\begin{aligned} I_1(t) &= -\sqrt{m\left(K_1^2(0) - K_1^2\right)} \\ &and \\ I_2(t) &= -\sqrt{m\left(K_2^2(0) - K_2^2\right)} \end{aligned}\right\}$$

which leads to

$$\frac{dK(t)}{dt} = -\sqrt{m}\left(\sqrt{K_1^2(0) - K_1^2} + \sqrt{K_2^2(0) - K_2^2}\right) \tag{3.7}$$

It can be shown that

$$\sqrt{K_1^2(0) - K_1^2} + \sqrt{K_2^2(0) - K_2^2} = \sqrt{(K_1(0) + K_2(0))^2 - (K_1 + K_2)^2} \;)$$

so that (3.7) can be written as

$$\frac{dK(t)}{dt} = -\sqrt{m(K_o^2 - K^2)} \tag{3.8}$$

where

$$K_o = K_1(0) + K_2(0) and K = K_1 + K_2$$

Equation (3.8) is an ordinary differential equation which on solving yields,

$$I_1(t) + I_2(t) = -K(0)\sqrt{m}\sin(\sqrt{m})t \tag{3.9}$$

where

K (0) = K_1 (0) +K_2 (0)

Rearranging (3.8) leads to the total increase in capital in terms of the invested capital $I_1(t)$ and $I_2(t)$ at initial time t=0, as

$$K^2 = K_o^2 - \frac{2}{m}\left(I_1^2 + I_2^2\right)$$

or

$$K = \sqrt{K_o^2 - \frac{2}{m}\left(I_1^2 + I_2^2\right)} \ ,$$

where $K_o = K_1(0) + K_2(0)$

Further, rearranging (3.9), leads to the period or time **t** that it takes to acquire the increased capital as

$$t = \frac{1}{\sqrt{m}} \sin^{-1}\left(\frac{I_1 + I_2}{-K(0)\sqrt{m}} \right)$$

In general, suppose now that $I_1, I_2, ..., I_r$ are the r capital amounts invested to lead to $K_1, K_2, ..., K_r$ increased capital amounts. A general model that would relate the system is obtained as

$$I_1^2 + I_2^2 + ... + I_r^2 = \frac{m}{r}\left(K_o^2 - K^2\right)$$

where

$K_o = K_1(0) + K_2(0) + ... + K_r(0) and K = K_1 + K_2 + ... + K_r$, which is the total increased capital and we only consider the total apart from its components. The relation (3 .8) can be written, using the summation symbol, as

$$\sum_{i=1}^{r} I_i^2 = \frac{m}{r}\left[\left(\sum_{i=1}^{r} K_i(0)\right)^2 - K^2\right] \tag{3.10}$$

From (3.10), we can obtain the total increased capital when r – capitals are invested at time t_o .Rearranging equation (3.10) leads to

$$K = \sqrt{\left(\sum_{i=1}^{r} K_i(0)\right)^2 - \frac{r}{m}\sum_{i=1}^{r} I_i^2}\ , \tag{3.11}$$

which provides a simpler relation for the total increased capital. The number **m** is a constant of proportionality between the rate of change in investment and rate of change investment and the deviation from the equilibrium capital. It depicts the appreciation factor for the investment. From (3.9) the total investment at time t for r – capitals invested is obtained as

$$I_1(t) + I_2(t) + ... + I_r(t) = -K(0)\sqrt{m}\sin\left(\sqrt{m}\right)t$$

where

$$K(0) = K_1(0) + K_2(0) + ... + K_r(0)$$

and this can be written as;

$$\sum_{j=1}^{r} I_j(t) = -\sum_{j=1}^{r} K_j(0)\sqrt{m}\sin(\sqrt{m})t \tag{3.12}$$

The negative sign is chosen in order to ease the modification. We can obtain the time **t** it takes an investment to gain in capitals from (3.12) as,

$$t = \frac{1}{\sqrt{m}}\sin^{-1}\left(\frac{\sum_{j=1}^{r} I_j}{-\sqrt{m}\sum_{j=1}^{r} K_j(0)}\right), \tag{3.13}$$

This is much simpler and faster to solve as compared to the piecewise method, where **m** is a constant of proportionality.

Example 3.1

Consider a case where Ksh 8,000 and Ksh 10,000 are invested in two businesses A and B respectively. If initially at t = 0, there was some capitals Ksh 5,000 and Ksh 6,000 respectively. To obtain the increased capital after some time t years and the time it takes to reach the current combined capital (take m=-200), then using the models (3.11) and (3.13) leads to the following values.

a) By using equation (3.11),

$$K = \sqrt{(5000 + 6000)^2 - \frac{2}{-200}\left((8000)^2 + (10000)^2\right)}$$

$$= \sqrt{121000000 + 1640000} \; = \sqrt{122640000} \text{ =Ksh } 11{,}074.30$$

b) Now solving for K_1 and K_2 separately leads to

$$I_1^2 = m\left(K_1^2(0) - K_1^2\right)$$

which implies that,

$(8000)^2 = -200\left((5000)^2 - K_1^2\right)$; m=-200 since K<K(0)

so that

$K_1^2 = 320{,}000 + 25{,}000{,}000$=25,320,000

Hence $K_1 = \sqrt{25320000}$ =Ksh 5031.90

Similarly,

$$I_2^2 = m\left(K_2^2(0) - K_2^2\right)$$

which implies that,

$$(10000)^2 = -200\left((6000)^2 - K_2^2\right)$$

so that

$K_2^2 - 36000000 = 500000$ implying that

$K_2^2 = 36000000 + 500000$=3,650,000

Hence

$K_2 = \sqrt{36500000}$ =Ksh 6041.50

Thus the total increased capital is given as;

$$K = K_1 + K_2$$

= Ksh (5031.90 +6041.50) =Ksh 11, 073.40

Comparing the results above, they give relatively the same answers. We can see that the developed mathematical model in (3.11) is a simpler method in terms of computer time and accuracy.

By virtue of equation (3.12), we get

$$t = \frac{1}{\sqrt{-200}} \sin^{-1}\left\{ \frac{(8000+10000)}{-\sqrt{-200}(5000+6000)} \right\}$$

$$= \frac{1}{i\sqrt{200}}\left[\sin^{-1}\left(\frac{18000}{11000\sqrt{200}} \right) \right] \cdot \frac{1}{-i}$$

$$= \frac{1}{\sqrt{200}} \sin^{-1}(0.1157) = \frac{6.644485}{\sqrt{200}} = 0.469836 \text{ years}$$

or

t =5.6 months

Hence it would take an approximate duration of 5 months and 20 days for Ksh 8000 and Ksh 10,000 invested in Ksh 5000 and Ksh 6000 capitals respectively to achieve a total of Ksh 11,074.30 increment.

Another Illustration

Similarly, if a businessman wants to invest some capitals in four existing investments of Ksh 2,600, Ksh1,800, Ksh 2,000 and Ksh 3,000. Suppose that a capital of Ksh 3,000 is invested in the first investment, a capital of Ksh 2,000 in the second investment, a capital of Ksh 2,500 in the third investment, and a capital of Ksh 3,500 in the fourth investment. The businessman would want to know how much he would achieve after some time operation if the flow rate is 100 per month and is constant. Also after how long would he achieve the amount?

(i) a) By using equation (3.11),

$$I_1 = Ksh3000$$
$$I_2 = Ksh2000$$
$$I_3 = Ksh2500$$
$$I_4 = Ksh3500$$

and

$$K_1(0) = Ksh2600$$
$$K_2(0) = Ksh1800$$
$$K_3(0) = Ksh2000$$
$$K_4(0) = Ksh3000$$

Therefore,

$$K=\sqrt{(2600+1800+2000+3000)^2+\frac{4}{100}(3000^2+2000^2+2500^2+3500^2)}$$

$$=\sqrt{(9400)^2+\frac{1}{25}(31500000)}$$

$=\sqrt{89620000}$ =Ksh 9466.80, which is the required increased capital.

b) By the piecewise system,

$$(3000)^2 = -100(2600^2 - K_1^2)$$

Thus

$$K_1^2 = 90000 + 6760000 \quad =6850000$$

or

$$K_1 = \sqrt{6850000} \quad =\text{Ksh } 2617.25$$

$$I_2^2 = m\left(K_2^2(0) - K_2^2\right)$$

or

$$(2000)^2 = -100\left(1800^2 - K_2^2\right)$$

Similarly,

$$K_2^2 = 3240000 + 40000$$

$K_2 = \sqrt{3280000}$ =Ksh1811.10

$I_3^2 = m\left(K_3^2(0) - K_3^2\right)$

And thus

$K_3^2 - 4{,}000{,}000 = 62{,}500$

or

$K_3^2 = 4{,}000{,}000 + 62{,}500$ =4,062,500

so

$K_3 = \sqrt{4{,}062{,}500}$ =Ksh 2,015.60

$I_4^2 = m\left(K_4^2(0) - K_4^2\right)$

Hence

$(3{,}500)^2 = -100\left(3000^2 - K_4^2\right)$

or

$K_4^2 = 9{,}000{,}000 + 122{,}500$ =9,122,500

$K_4 = \sqrt{9{,}122{,}500}$ =Ksh 3,020.35

Hence the total increased capital would be

$K = K_1 + K_2 + K_3 + K_4$

=Ksh (2,617.25+1,811.10+2,015.60+3,020.35)=Ksh 9,464.30

The results obtained in (a) and (b) are comparatively close to each other.

(ii) Turning to the time it would take, we use the model in (3.13) as

$$\sum_{j=1}^{4} I_j = Ksh\left(3{,}000 + 2{,}000 + 2{,}500 + 3{,}500\right) = Ksh11{,}000$$

and

$$\sum_{j=1}^{4} K_j(0) = Ksh(2{,}600 + 1{,}800 + 2{,}000 + 3{,}000) = Ksh9{,}400$$

Therefore, $t = \dfrac{1}{\sqrt{-100}} \sin^{-1}\left\{\dfrac{11{,}000}{-9{,}400\sqrt{-100}}\right\}$

$$= \frac{1}{10i}\left[\sin^{-1}(0.1170)\right]\frac{1}{-i} = \frac{6.72}{10} = 0.672 months \sim 20 days$$

Thus, it would take approximately 20 days to achieve an amount of Ksh 9,466.80 increment.

The total increased capital when $I_1, I_2, ..., I_r$ are invested in the investments with $K_1(0), K_2(0), ..., K_r(0)$ current capitals at time t= 0 is given by equation, mathematical model in (3.9). The time duration for which the increased capital is attained is obtained from the equation (3.13).

3.2 Advantages and Disadvantages

The mathematical models developed in (3.11) and (3.13) have the following advantages: The models offer a direct way of obtaining the total capital investment as compared to the piecewise system, which is time consuming. The models save time and computer memory space. Lastly, the models are suitable to ease the work in calculations involving large figures.

These models, equations (3.11) and (3.13) have only one major limitation. The models give a total but not the components of the total. For instance the model (3.11) gives total increased capital **K** but does not tell the values of its components, that is $K_1, K_2, ..., K_r$ as pertains to each individual portfolio. Thus in practice, it would be essential to use (3.11) and (3.13) if the goal is to know the overall profit. Otherwise, if the profit from each portfolio is essential, then the piecewise method is appropriate. In practice, one would need both methods depending on the goals.

CHAPTER FOUR

MATHEMATICAL MODELS FOR THE COMPOUND AMOUNT

4.1 Introduction

When a sum of money is invested, it gains interest at a certain rate. For small savings accounts, one may want to pay in money from time to time .One may also want to withdraw cash at times.

Suppose that S_0 is invested at a compound interest rate of i% per unit amount per unit time and S_t is the amount after time t. The model depicting the amount at any time is given by the difference equation,

$$S_{t+1} = (1+i)S_t$$

which has the solution as

$$S_t = S_0 (1+i)^t \qquad (4.1)$$

This represents the amount in the account at any time **t**. The amount $(1+i)^t$ is referred to as the compounding value interest factor or future value factor. The relation given in (4.1) is what is widely used in business calculations to obtain the compound amount.

If the future value factor is expanded using the binomial expansion theorem, it leads to,

$$(1+i)^t = 1 + \frac{it}{1!} + \frac{t(t-1)i^2}{2!} + \frac{t(t-1)(t-2)i^3}{3!} + \ldots \qquad (4.2)$$

Now, making mathematical adjustment on the right hand side by varying the parameter t such that

$$\left.\begin{aligned} t &= \left(\frac{t'}{k}\right)^1, \\ t(t-1) &= \left(\frac{t'}{k}\right)^2, \\ t(t-1)(t-2) &= \left(\frac{t'}{k}\right)^3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad (4.3)$$

k represents a constant rate over which the parameter **t** is varied

Substituting equations (4.3) into the equation (4.2), leads to we;

$$(1+i)^t = 1 + \left(\frac{it}{k}\right) + \frac{1}{2!}\left(\frac{it}{k}\right)^2 + \frac{1}{3!}\left(\frac{it}{k}\right)^3 + \ldots$$

which can also be written as,

$$(1+i)^t = \sum_{n=0}^{\infty}\left(\frac{it}{k}\right)^n \cdot \frac{1}{n!};\ for\, 0! = 1 \qquad (4.4)$$

The quantity given in (4.4) is an exponential function such that

$$\sum_{n=0}^{\infty}\left(\frac{it}{k}\right)^n \cdot \frac{1}{n!} = \ell^{\frac{it}{k}} \qquad (4.5)$$

By virtue of equations (4.5) and (4.4), equation (4.1) can be reduced to the form

$$S_t = S_o \ell^{\frac{it}{k}} \qquad (4.6)$$

where number **k** is a constant over the whole period at a given interest rate. Thus, k depends only on the interest rate. The values of k for given interest rates are given in the table below:

TABLE 4.1: Values of k in relation to rates.

Range of i	1.0-1.9	2.0-2.9	3.0-3.9	4.0-4.9	5.0-5.9	6.0-6.9	7.0-7.9	8.0-8.9
Values of k	1.005	1.010	1.015	1.020	1.025	1.030	1.035	1.040

9.0-9.9	10-10.9	11-11.9	12-12.9	13-13.9	14-14.9	15-15.9	16-16.9	17-17.9
1.045	1.050	1.055	1.060	1.065	1.070	1.075	1.080	1.085

18-18.9	19-19.9	20-20.9	21-21.9	22-22.9	23-23.9	24-24.9
1.090	1.095	1.100	1.105	1.110	1.115	1.120

The relation given in (4.6) compares closely with other methods. As an example, suppose that one need to find to the future value of $1000 invested at 8.8% compounded for 5.5 years, then the compound amounts based on the various methods are as follows:

a) Using (4.6), k=1.040 from table 4.1 and thus

$$S_{5.5} = \$1000\ell^{\frac{0.088\times5.5}{1.040}} = \$1000(1.5926) = \$1,592.60$$

b) By the compound amount formula, given in (4.1)

$$S_{5.5} = \$1000(1+0.088)^{5.5} = \$1000(1.5902) = \$1,590.20$$

c) Using the relation of Khoury and Parsons (1981), we have

CIF=1.592 where CIF is the compound interest factor, and thus the future value or compound amount is obtained as

$$FV^{n} = \$A(CIF)$$

Therefore,

$$FV^{5.5} = \$1000(1.592) \quad = \$1,592.00$$

The three results obtained above are relatively the same. The accuracy depends entirely on the nature of the model and underlying assumptions of the model.

As another example, suppose a businessman deposits Ksh89,500 in a bank that offers a compound interest of 14%per annum then the amount of money that he would have in the account for 5 years can be obtained by the various methods as follows:

a) By the model (4.6), k=1.070 from table 4.1 and thus

$$S_5 = ksh89,500\ell^{\frac{0.14\times5}{1.070}} = ksh89,500(1.9236) \quad \text{=Ksh172,163.40}$$

b) Using the compound interest formula given in (4.1)

$$S_5 = ksh89,500(1+0.14)^5$$

or $$S_5 = ksh89,500(1.14)^5 \text{=Ksh89,500(1.9254)} \quad \text{=Ksh172,234.60}$$

The two results obtained above are relatively close.

When the interest is compounded over periods other than a year, then the mathematical model is adjusted accordingly. If the amount is compounded **m** periods in a year then the interest rate becomes **m** times the rate, the time also changes to **t** times **m**.

Generally, the future value of a sum S_0 invested at I% compounded **m** times over a period of **t** years is obtained as

$$S_t = S_0 \ell^{\left(\frac{i}{m}\right)\frac{mt}{k}} \quad \text{or} \quad S_t = S_0 \ell^{\frac{i'mt}{k}} \tag{4.7}$$

where $i' = \frac{i}{m}$

If for example $1000 is invested at 8% compounded quarterly for 2 years, then

a) Using the model (4.7), $i' = \frac{8}{4}$=2%, k=1.010, t=8. Hence we have

$$S_2 = \$1000\ell^{\frac{0.02\times 8}{1.010}} = \$1000(1.172) = \$1,172$$

b) By the relation in (4.1), we have

$$FV^n = A(1+\frac{i}{m})^{mn}$$

$$FV^2 = \$1000(1+\frac{0.08}{4})^{4\times 2} = \$1000\,(1+0.02)^8 = \$1000(1.02)^8 = \$1000(1.172)$$
$$= \$1,172$$

The two methods give similar values.

Considering a case where an equal sum of money is invested in different firms, such that each sum of money gains interest at a certain rate. Then if S_o is invested in three different banks at compound interest rates of i_1, i_2, i_3 respectively per unit time, then each would have an amount

$$\left.\begin{aligned} S_1(t) &= S_o \sum_{n=0}^{\infty} \left(\frac{i_1 t}{k}\right)^n . \frac{1}{n!} \\ S_2(t) &= S_o \sum_{n=0}^{\infty} \left(\frac{i_2 t}{k}\right)^n . \frac{1}{n!} \\ S_3(t) &= S_o \sum_{n=0}^{\infty} \left(\frac{i_3 t}{k}\right)^n . \frac{1}{n!} \end{aligned}\right\} \qquad (4.8)$$

From equation (4.8), we can be able to obtain a generalised amount after time t,

$$S_t = S_1 + S_2 + S_3$$

$$= S_o \left[\sum_{n=0}^{\infty} \left(\frac{i_1 t}{k}\right)^n . \frac{1}{n!} + \sum_{n=0}^{\infty} \left(\frac{i_2 t}{k}\right)^n . \frac{1}{n!} + \sum \left(\frac{i_3 t}{k}\right)^n . \frac{1}{n!} \right]$$

or $$S_t = S_o \left[2 + \sum_{n=0}^{\infty} \left((i_1 + i_2 + i_3) \frac{t}{k} \right)^n . \frac{1}{n!} \right] \qquad (4.9)$$

Using equation (4.5), equation (4.9) becomes

$$S_t = S_o \left[2 + \ell^{(i_1 + i_2 + i_3)\frac{t}{k}} \right]$$

where **k** is obtained from table 4.1 from the column containing the sum of the interest rates plus the greatest rate.

Suppose a sum of money S_o is invested equally in r firms at, interest rates $i_1, i_2, ..., i_r$ respectively, then the total amount S_t after time t will be

$$S_t = S_o\left[(r-1)+\ell^{\left(\frac{i_1+i_2+\ldots+i_r}{k}\right)t}\right] \quad (4.10)$$

As an example if an equal sum of Ksh6000 is invested for 2 years in banks A,B and C that offer compound interest rates 3%,4% and 2% per year respectively. Then

a) Applying (4.10), r=3, r-1=2, k=1.065. Thus,

$$S_2 = ksh6000\left[2+\ell^{\frac{0.09\times 2}{1.065}}\right]$$ =Ksh6000 (3.1841) =Ksh19,104.60

b) By the piecewise method;

Amount at bank A is

$S_1 = ksh6000(1+0.03)^2$ =Ksh6000 $(1.03)^2$ =Ksh6000 (1.0609) =Ksh6,365.40

Amount at bank B is

$S_2 = ksh6000(1+0.04)^2$ =Ksh6000 $(1.04)^2$ =Ksh6000 (1.0816) =Ksh6,489.60

Amount at bank C is

$S_3 = ksh6000(1+0.02)^2 = ksh6000(1.02)^2$ =Ksh6000 (1.040) =Ksh6,242.40

Therefore, total amount,

$S = S_1 + S_2 + S_3$ =Ksh (6365.40+6489.60+6242.40) or

S=Ksh19,097.40

The results in (a) and (b) show a close relationship and hence the mathematical model (4.10) holds for all k>0.

The mathematical model (4.10) can also be used in the case of the continuous compounding. We simply drop k, unlike for the discrete case, that is

$$S_t = S_o\left[(r-1) + \ell^{(i_1+i_2+i_3)t}\right]$$

4.2 Advantages and Disadvantages

The mathematical model given in equation (4.10) has the following advantages over other models or methods: The model is easy to use and its application requires little computer time and thus it saves time.. The model can be used easily when an amount is distributed over various interest rates. The mathematical model developed, equation (4.10) has one limitation in the sense that it cannot be used to calculate the interest rate or the time. Therefore, it only helps in calculating amount. The mathematical model (4.10) is a generalisation of the model given by (4.1).

CHAPTER FIVE

MODELS FOR ANNUITIES

5.1 Introduction

An annuity is a series of identical cash flows that are expected to occur at each period for a specified number of periods.

The future value of an annuity can be computed by applying the relation given as $FV_n = PV(1+i)^n$ to each payment and then summing the individual values to get the total, where FV is the future value of the annuity. That is,

$$FVA_n = CF\sum_{t=0}^{n-1}(1+i)^t \tag{5.1}$$

Using

$$\sum_{t=0}^{n-1}(1+i)^t = \frac{(1+i)^n - 1}{i}$$, leads to,

$$FVA_n = CF\left[\frac{(1+i)^n - 1}{i}\right] \tag{5.2}$$

where; FVA_n is the future value of the annuity after n periods and CF is the amount deposited at each period, which is constant. If a person borrows an amount at a compound interest rate i and wants to pay back in equal instalments say, a total of R, with the first amount being paid at the end of the first year. Then if S_t is the amount due at the end of **t** years, then we have the mathematical model as

$$S_{t+1} = S_t + iS_t - R \qquad \text{or} \qquad S_{t+1} = (1+i)S_t - R$$

which is a linear difference equation that can be solved to give

$$S_t = \left(S_o - \frac{R}{i}\right)(1+i)^t + \frac{R}{i} = S_t = S_o(1+i)^t - R\left[\frac{(1+i)^t - 1}{i}\right] \tag{5.3}$$

Consider a case where an amount **R** is deposited at the end of even periods in a bank and S_t is the amount at the end of **n** periods. Then, the mathematical model is given by the linear difference equation,

$$S_{t+1} = S_t(1+i) + R \tag{5.4}$$

which can be solved by linear difference method to give,

$$S_n = R\left[\frac{(1+i)^n - 1}{i}\right]; \qquad \text{for} \quad S_o = 0 \tag{5.5}$$

5.2 Alternative Relation

We consider two major areas with their respective general mathematical models.

5.2.1 Depositing:

Consider a case where an amount R is deposited at the end of even periods in a bank and S_t is the amount at the end of **t** periods. Then applying the future value model (5.4) to each payment and then summing the individual values to get the total, that is

$$S_t = R\sum_{t=0}^{n-1} \ell^{\frac{it}{k}}$$

or

$$S_t = R\left[1 + \ell^{\frac{i}{k}} + \ell^{\frac{2i}{k}} + \ldots + \ell^{\frac{(t-1)i}{k}}\right] \tag{5.6}$$

Multiplying (5.6) by $\ell^{\frac{i}{k}}$ on both sides, we obtain

$$\ell^{\frac{i}{k}} S_t = R\left[\ell^{\frac{i}{k}} + \ell^{\frac{2i}{k}} + \ell^{\frac{3i}{k}} + \ldots + \ell^{\frac{(t-1)i}{k}} + \ell^{\frac{it}{k}}\right] \tag{5.7}$$

Now, subtracting (5.6) from (5.7) we obtain

$$\left(\ell^{\frac{i}{k}} - 1\right) S_t = R\left(\ell^{\frac{it}{k}} - 1\right)$$

or

$$S_t = R\left[\frac{\ell^{\frac{it}{k}} - 1}{\ell^{\frac{i}{k}} - 1}\right] \tag{5.8}$$

which is the required amount at the end of time **t** when a constant amount **R** is deposited at the end of each period, for k>o.

Example 5.2.1

If an amount Ksh4000 is equally invested at the end of each year in a bank that offers 5% interest compounded yearly. Determine the amount in the account after 10 years.

a) Using the model (5.8), k=1.025 from table 4.1.Therefore,

$$S_{10} = ksh4000\left[\frac{\ell^{\frac{0.05\times10}{1.025}} - 1}{\ell^{\frac{0.05}{1.025}} - 1}\right] = \text{Ksh } 4000(12.57729606) = \text{Ksh } 50{,}309.20$$

b) Applying the usual method, from equation (5.5) we obtain

$$S_n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$S_{10} = ksh4000\left[\frac{(1+0.05)^{10} - 1}{0.05}\right] = \text{Ksh } 4000\left[\frac{(1.05)^{10} - 1}{0.05}\right] = \text{Ksh } 4000\left(\frac{0.628894627}{0.05}\right)$$

=Ksh 4000(12.57789254) =Ksh 50,311.60

The two results in (a) and (b) give close values. Hence, the model developed, as in equation (5.8) holds for k>0.

If an amount **R** is deposited at the end of even periods in **r** banks at compound rates $i_1, i_2, \ldots, i_r$ respectively per year and S_t ***is the*** total amount at the end of **t** periods. Then model (5.8) generally becomes,

$$S_t = R\left[\frac{\ell^{\frac{(i_1+i_2+\ldots+i_r)t}{k}} - 1}{\ell^{(i+i_2+\ldots+i_r)\frac{1}{k}} - 1}\right] \tag{5.9}$$

where **k** is obtained from table 4.1 by checking the column of interest rates plus the largest rate.

5.2.2 Borrowing:

Let us take a case where a person borrows an amount at a compound interest rate i and wants to pay back in equal investments say a total of **R** with first amount being paid at the end of the first year. Let S_t be the amount due at the end of **t** years, then we have

$S_{t+1} = S_t + iS_t - R$ or $S_{t+1} = \ell^{\frac{i}{k}} S_t - R$,

which can be solved to give;

$$S_t = S_o \ell^{\frac{it}{k}} - R\left[\frac{\ell^{\frac{it}{k}} - 1}{\ell^{\frac{i}{k}} - 1}\right] \tag{5.10}$$

If the amount is paid back in n-years, then equation (5.10) reduces to;

$$R = S_0 \frac{\left(\ell^{\frac{i}{k}} - 1\right)}{\left(1 - \ell^{-\frac{it}{k}}\right)} \tag{5.11}$$

where k>0 holds from table 4.1.

Example 5.2.2

What would be the monthly payment on $ 1500 car loan borrowed for 1 year at 12% p.a.

a) Applying the model developed (5.11), i=0.01, k=1.005(from table 4.1), n=12.

Hence,

$$R = \$1500\left[\frac{\ell^{\frac{0.01}{1.005}} - 1}{1 -- \ell^{\frac{-0.01\times12}{1.005}}}\right] = \$1500\left(\frac{0.009999917}{0.1125499}\right) = \$1500(0.0888487) = \$133.27$$

b) Considering equation (5.3) when S_t=0 then,

$$R = \frac{S_o i}{1-(1+i)^{-n}} = \$\frac{1500(0.01)}{1-(1+0.01)^{-12}} = \$\frac{1500(0.01)}{(1-0.887449225)} = \$\frac{15}{0.112550774}$$

=$133.2

c) Using the relation by Sarkis and Torrence (1981), we have

$$PV^{n}_{a,I} = a\left(PVDF^{12}_{a,I}\right) = \$1500$$

Thus

$$a = \$\frac{1500}{PVDF^{12}_{a,I}} = \$133.27,$$

where the value of $PVDF^{12}_{a,I}$ is given in the text.

Therefore, the three approaches give the same results and hence, the developed model in (5.11) holds for k>0.

Generally, if an amount is borrowed from different firms say **r** at compound interest rates $i_1, i_2, ..., i_r$ respectively and is to be repaid in equal instalments say a total of **R** to each firm with first amount being paid at the end of the first year and S_t is the amount due at end of **t** years, then,

$$S_t = S_o \ell^{\frac{(i_1+i_2+...+i_r)t}{k}} - R\left[\frac{\ell^{\frac{(i_1+i_2+...i_r)t}{k}} - 1}{\ell^{\frac{(i_1+i_2+...+i_r)}{k}} - 1}\right] \tag{5.12}$$

If the total amount is paid back in n-years, then $S_n = 0$. and hence equation (5.12) reduces to

$$R = S_o\left[\frac{\ell^{\frac{(i_1+i_2+...+i_r)}{k}} - 1}{1-\ell^{\frac{-(i_1+i_2+...+i_r)n}{k}}}\right], \tag{5.13}$$

as the amount to be paid back in each firm at the end of each period.

5.3 Advantages and Disadvantages

The mathematical models in (5.10), (5.11), (5.12) and (5.13) have the following advantages over other models: First the models can be used easily when an amount is distributed over various interest rates; they require little space to manipulate; and they give solutions fast hence saving time and they are easy to use.

The above mathematical models have one limitation in that they cannot be used to calculate the interest rate or the time. Therefore, they only help in calculating the amount.

FUTURE WORK

My further research will constitute a consideration of unconstrained profit maximization based on the following constraints:

1. Cash availability
2. Required earnings per share
3. Environmental factors.

I will incorporate the net present value and the internal rate of return maximization models using the linear programming method.

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