COFINITE TOPOLOGICAL MANIFOLDS AND INVARIANCE OF TOPOLOGICAL PROPERTIES WITH RESPECT TO ALMOST CONTINUOUS FUNCTIONS

Were Hezron Saka

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DECLARATION AND RECOMMENDATION

DECLARATION

This thesis is my original work and has not been submitted in part or whole for an award in any university.

Mr. Were H. S.

SM12/3024/11

Signature: ________________________________ Date: __________________

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This thesis has been submitted with our approval as supervisors for examination according to Egerton University regulations.

Prof. Sogomo K. C.

Egerton University

Signature: _______________________________ Date: ____________________

Dr. Gichuki M. N.

Egerton University

Signature: _______________________________ Date: ____________________

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DEDICATION

This dissertation is dedicated to my wife Rhoda A. Saka, my mother Yunia A. Were and my late father Samson Fred Were all of whom I attribute my success in life to their tireless efforts in my upbringing and great support in my academics.

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ABSTRACT

Manifolds are generalization of curves and surfaces to arbitrary higher dimensions. They are of many kinds, one of them being topological manifolds. The main feature common to manifolds is that every point of the space is in one to one correspondence with a point in another space. Hausdorff manifolds have been developed on infinite dimensional spaces such as Banach spaces and Fréchet spaces. Topological properties of non-Hausdorff manifolds have been studied and the notion of compatible apparition points have been introduced for such non-Hausdorff manifolds. Among many generalizations of the notion of continuity, almost continuous functions have been used on both T_1 and T_2 – spaces. In such spaces, the properties and characterization of almost continuous functions have been studied and interesting results have been obtained especially with the class of T_1 – spaces. Invariance and inverse invariance of some topological properties have been investigated with respect to almost continuous functions and continuous functions. Manifolds have not been modeled on a T_1 – space having cofinite topology. Invariance of topological properties such as compactness and its other notions have not been investigated from an infinite cofinite space to the Euclidean space with respect to almost continuous functions. This study has therefore investigated the invariance of these topological properties from the cofinite space to the Euclidean $\mathbb{R}^n$ spaces with respect to almost continuous functions. Differentiable manifolds are among the most fundamental notions of modern mathematics as it is the cornerstone of modern mathematical science. The study has also obtained a cofinite manifold where almost continuous functions have been used as maps. *Pseudoderivative* on almost continuous functions has also been defined in this study and some of its properties have been stated. This has facilitated the development of *pseudodifferentiable* cofinite manifolds. Manifolds, especially the differentiable manifolds have applications in survey and physics. In physics, the applications are found in mechanics and electromagnetics where boundary value problems are solved in mesh generation.

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LIST OF NOTATIONS

$\subset$	Subset of
$\in$	Contained in/an element of
$\notin$	Not an element of
$\forall$	For all
$f\|_A$	Restriction of function f to A , a subset (subspace) of a given set (space)
f^{-1}	Inverse of a function f
$\cup$	Union
$\cap$	Intersection
A'	Complement of a set A
$\emptyset$	Empty set
$\mathcal{T}$	Topology on a space
$\exists$	There exist
$(\overline{A})$ or $Cl(A)$	Closure of A
$\longrightarrow$	Domain range relation
$\mathbb{R}^n$	n dimensional Euclidean space
C	Cofinite space
$\setminus$	Relative complement
$G(f)$	Graph of a function f
$\phi(x_0)$	Separable space with cofinite topology
$\psi(x_0)$	Separable space with cofinite topology
U, V	Convex balanced, open neighborhoods of zero in $\phi(x_0)$ and $\psi(x_0)$ respectively
β, γ	Systems of bounded, convex, balanced and closed sets in $\phi(x_0)$ and $\psi(x_0)$ respectively
B, C	Sets from β and γ respectively
$\beta_{\mathcal{B}}, \beta_C$	Systems of all non-empty convex balanced and closed bounded (incase of $\beta_{\mathcal{B}}$) and compact (incase of β_C) sets in $\phi(x_0)$
$\gamma_{\mathcal{B}}, \gamma_C$	Systems of all non-empty convex balanced and closed bounded (incase of $\gamma_{\mathcal{B}}$) and compact (incase of γ_C) sets in $\psi(x_0)$

CHAPTER ONE

INTRODUCTION

1.1 Background information

Manifolds generalize the notion of curves and surfaces in two and three dimensions to higher dimensions. Examples of manifolds start with open domains in the Euclidean space $\mathbb{R}^n$. A manifold has been looked at in its broadest sense as a topological space locally homeomorphic to the Euclidean space $\mathbb{R}^n$ of a fixed dimension without assuming the Hausdorff separation axiom (Baillif and Gabard, 2008). The definition of a topological manifold M of dimension n which is based on the properties of Hausdorffness (for every pair of distinct points $p, q \in M$ there exists a disjoint open subsets $U, V \subset M$ such that $p \in U$ and $q \in V$), second countability (there exists a countable basis for the topology of M) and that M is locally Euclidean of dimension n (every point in the manifold M has a neighborhood that is homeomophic to an open subset of $\mathbb{R}^n$, that is $\Phi : U \subset M \longrightarrow \mathfrak{U} \subset \mathbb{R}^n$) was given by Lee (2000). This definition of topological manifold has also been shown in much of the mathematical literature. This shows that topological manifolds have been defined on Hausdorff spaces and a continuous function have been used from these spaces to Euclidean $\mathbb{R}^n$ spaces allowing manifolds to behave in the same way Euclidean spaces do. This holds since continuity is defined on T_2 – spaces and above and its existence is also guaranteed by Urysohn's lemma. The concept of Manifolds enables us to connect non-Euclidean spaces to some similar objects in the Euclidean spaces. Intuitively the charts give local coordinates on the manifolds and allow us to treat it like Euclidean space. Any local or global object defined on the manifold including functions, vectors and tensors will be linked back to Euclidean spaces through charts and atlases. A non - Hausdorff manifold was defined as a topological space which has a countable base of open sets and locally Euclidean of dimension n (Kent *et al*., 2009). The article stated that one can easily show that every non-Hausdorff manifold is a T_1 – space since every point in such a space has a Euclidean neighborhood showing that manifold can be defined on T_1 – spaces. This relaxed the idea of a Hausdorff property, being independent of other conditions from the standard definition of topological manifold.

Among many generalization of notion of continuity, almost continuous functions have been defined on both Hausdorff spaces and T_1 – spaces. For the product space $X \times Y$ which is a completely normal Hausdorff space, where X and Y are topological spaces and X is connected, it was shown by Stallings (1959) that if $f: X \longrightarrow Y$ is an almost continuous function then the graph $G(f)$ is connected. Given that $A \subset X$ and $f : X \longrightarrow A$ is an almost continuous function in the sense of Singal and Singal (1968) which is a retraction of X onto A, then A is a closed subset of X provided that X is a Hausdorff space (Long and Carnahan, 1973). The properties of almost continuous functions were also studied where one of the theorems in the article defined almost continuous function on a first countable T_1 – space X to another first countable space Y (Long and McGehee, 1970). Other properties such as the composition of an almost continuous function $f: X \longrightarrow Y$ and a continuous function $g : Y \to Z$, that is $g \circ f : X \to Z$ being almost continuous, was shown by Stallings (1959), where X, Y and Z were considered topological spaces. Invariance and inverse invariance of some topological properties with respect to continuous functions and almost continuous functions have been studied (Gichuki, 1996). On separable locally convex topological vector spaces, one variant of the inverse function theorem have been developed where D_{β_c,γ_c} derivative at a point in a given open subset of a locally convex space have been shown (Sukhinin and Sogomo, 1985).

1.2 Statement of the problem

Topological manifolds have been modeled on various spaces such as convenient vector spaces, Hilbert spaces, Banach spaces and Fréchet spaces. These manifolds have been defined based on a Hausdorff property on such spaces. But manifolds have not been developed on T_1 – spaces having cofinite topology (cofinite spaces). Invariance of topological properties such as compactness and its other notions have not been investigated from an infinite cofinite space to the Euclidean $\mathbb{R}^n$ space with respect to almost continuous functions. Therefore the study have investigated the invariance of compactness and its other notions with respect to almost continuous functions from the cofinite space which is a T_1 – space to the Euclidean $\mathbb{R}^n$ space. A *pseudoderivative* on almost continuous functions has also been defined and its properties stated which has facilitated the development of cofinite topological *pseudodifferentiable* manifolds.

1.3 Objectives

1.3.1 General objective

To investigate the invariance of topological properties with respect to almost continuous functions from cofinite space to the Euclidean R^n spaces and develop a cofinite topological manifold using almost continuous functions as maps.

1.3.2 Specific objectives

1. To determine the invariance of topological properties from the T_1 – space with cofinite topology (cofinite space) to the Euclidean $\mathbb{R}^n$ space with respect to almost continuous functions.
2. To define *pseudoderivative* on almost continuous functions and find some of its properties.
3. To develop topological manifold on a T_1 – space with cofinite topology (cofinite space) using almost continuous functions as maps.

1.4 Justification

Cofinite topology as the coarsest topology satisfying the T_1 – axiom, for which every singleton set is closed, is depicted on finite spaces as well as infinite spaces. When the space X is infinite then the topology on it will not be T_2, regular or normal as no two nonempty open sets are disjoint. Realization of invariance of topological properties in such kind of spaces with respect to almost continuous functions to Euclidean $\mathbb{R}^n$ spaces may lead to generalization of the existing results on topological manifolds defined on Hausdorff spaces. It will also be a means of studying some topological properties that could not be easily studied in Euclidean space directly. Differentiable manifolds have been extensively studied due to their many applications in mechanics and electromagnetics. Generalization to *pseudodifferentiable* cofinite manifolds has however not been done yet and this could enhance the applications of manifolds in mechanics and electromagnetics. Modeling of topological manifolds on cofinite spaces will enable the properties of cofinite topology such as those of subspaces, separation and compactness and its other notions to be more useful on manifolds. Properties such as points of a T_1 – space which are closed sets will also be useful in the modeled topological manifolds.

CHAPTER TWO
LITERATURE REVIEW

2.1 Overview of literature

Manifolds have been studied in both low and high dimensions. The classification theorem of two dimensional manifolds is known, the studies of three dimensional manifolds were pioneered by Poincaré in the 20th century and in the 1980s, Michael Freedman brought in the forefront of mathematical research the four dimensional manifolds (Lee, 2000). Manifolds are of many kinds, one of them being topological manifold. Topological manifold have been defined based on second countability, Hausdorffness and locally Euclidean spaces of dimension n. Continuity has been defined from topological manifold to Euclidean $\mathbb{R}^n$ spaces. On a topological manifold, a superposition operator in the space of vector valued, bounded and continuous functions have been considered. The acting conditions and criteria of continuity and compactness have also been established (Dronka, 2010). In 1950s and 1960s, a broad definition of a manifold was given which omitted the point set axioms and allowed higher cardinalities and non-Hausdorff manifolds to be modeled. It also omitted finite dimension which allowed structures such as Hilbert manifolds to be modeled on Hilbert spaces, Banach manifolds to be modeled on Banach spaces and Fréchet manifolds to be modeled on Fréchet spaces. Manifolds have been modeled on convenient vector spaces, which are locally convex vector spaces that are C^∞- complete (Kriegl and Michor, 1997). The topology of non-Hausdorff manifolds was investigated and applications to foliations where manifolds were necessarily T_1 – spaces but not guaranteed to be Hausdorff were given (Gartside *et al.*, 2008). Some topological properties of non-Hausdorff manifolds were reviewed and the notion of compatible apparition points for the non-Hausdorff manifolds were introduced where the properties of these points were studied (Kent *et al.*, 2009).

The concept of almost continuous function was studied for real valued functions on Euclidean spaces by Blumberg (1922). Almost continuous function had been defined differently (Stallings, 1959; Frolik, 1961; Husain, 1966; Singal and Singal, 1968). These concepts of almost continuous functions had been shown using examples that they are independent of each other (Mamata, 1971). Almost continuity generalizes the notion of continuity and every continuous function is an almost continuous function even though the reciprocal may not hold. To visualize this, consider two topological spaces X and Y, each being the set of all real numbers with the usual topology where the open sets are taken to be the open intervals in the real line, then the function $f : X \to Y$ defined by: $f(x) = \begin{cases} \sin\frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$ is an almost continuous function that is not continuous at $x = 0$. This is so since this function oscillates near the point $x = 0$, hence its limit cannot exist at that point implying that the function cannot be continuous at the point $x = 0$. Almost continuous mapping have been introduced in several spaces and its properties and characterization have been studied. Some of the properties and several results concerning almost continuous functions have been studied and proved (Long and McGehee, 1970). Invariance and inverse invariance of some topological properties with respect to continuous functions and almost continuous functions have been studied and discussed by Gichuki (1996). Since all continuous functions are subsets of almost continuous functions, a property that is not preserved by continuous functions cannot be preserved by almost continuous functions. It had been established that among the class of $T_1 -$ spaces, invariance of topological properties with respect to continuous bijections (one to one and onto continuous functions) implies invariance of the same topological properties with respect to almost continuous bijections, and that if a property P is invariant of continuous functions, it must be invariant of continuous bijections and hence invariant of almost continuous bijections in the class of $T_1 -$ spaces (Gichuki, 1996). A remark was made that most of the interesting results with almost continuous functions are obtained with the class of $T_1 -$ spaces (Naimpally, 1966). The study of almost continuous path connected spaces showed that they are connected spaces (Herrington and Long, 1981). Almost continuous function on almost regular spaces were introduced by Shin and Lee (1983), where they studied the conditions of mappings under which the image of almost locally connected space is almost locally connected space. Some generalization of almost continuity in the sense of Singal and Singal (1968) were obtained by Noiri (1989) and showed that every nearly almost open and almost weakly continuous function is almost continuous in the sense of Husain (1966).

The ordinary derivative at a point x for any real valued function $f : \mathbb{R} \to \mathbb{R}$ has been defined in classical calculus and this has been obtained as a number. A generalization of this ordinary derivative of a real valued function of a single real variable to the case of vector valued functions of multiple real variables have been developed. This is known as Fréchet derivative. It extends the concept of the derivative to operator in general normed spaces such as infinite dimensional function spaces and gives the best local linear approximation in the neighborhood of a point x (Griffel, 1985). On separable locally convex topological vector spaces, one variant of the inverse function theorem has been developed where D_{β_c,γ_c} derivative at a point in a given open subset of a locally convex space have been shown (Sukhinin and Sogomo, 1985). Various notion of compactness such as countably compactness, limit point compactness, sequentially compactness and pseudocompactness have been studied by Yu (2012). The article looked at their useful properties and their relations on arbitrary topological spaces as well as on metric spaces.

2.2 Cofinite space and almost continuous functions

A subset U of an infinite set X is defined to be open if $U = \emptyset$ or $X \setminus U$ is finite. This defines a topology on X called a cofinite topology. This infinite set X together with the cofinite topology on it is known as the cofinite space (Deshpande, 1990). Properties of cofinite space such as compactness, connectedness, sequential compactness, separability and limit point compactness have triggered the investigation of invariance of topological properties with respect to almost continuous functions to the Euclidean $\mathbb{R}^n$ spaces. The culmination on this space is therefore achieved when a topological manifold is modeled using almost continuous functions as maps since a topology can be defined on it.

Almost continuous functions as stated by Prakash and Srivastava (1977) were defined differently by Stallings (1959), Frolik (1961), Husain (1966) and Singal and Singal (1968) as follows:

Let $f: X \longrightarrow Y$ be a function from a set X into the set Y. The set $G(f) = \{(x, f(x))\}$ is called the graph of f. Given topological spaces X and Y, a function $f: X \longrightarrow Y$ is said to be almost continuous if for each open set U in $X \times Y$ containing $G(f)$, there exists a continuous function $g: X \longrightarrow Y$ such that $G(g) \subseteq U$ (Stallings, 1959). According to Gichuki (1996), this definition of almost continuous functions implies that given any two neighborhoods U_1 and U_2 of $G(f)$ in $X \times Y$, there exists a function $g: X \longrightarrow Y$ such that $G(g) \subseteq U_1 \cap U_2$ and that a similar results holds when one consider any finite number of neighborhoods of $G(f)$. A mapping $f : X \longrightarrow Y$ is said to be almost continuous if for every open subset $V \subset Y$, $f^{-1}(V) \subset Cl(Int\, f^{-1}(V))$ by Frolik (1961). A different version of this concept of almost continuous function was also given by Husain (1966) as a function $f : X \longrightarrow Y$ where X and Y are topological spaces is almost continuous at $x \in X$ if for each open set $V \subset Y$ containing $f(x)$, $Cl(f^{-1}(V))$ is a neighborhood of x. If f is almost continuous at each point of X, then f is called almost continuous. Finally a definition given by Singal and Singal (1968) was that a mapping $f : X \longrightarrow Y$ is said to be almost continuous at a point $x \in X$ if for every neighborhood M of $f(x)$ there is a neighborhood N of x such that $f(N) \subset Int(Cl(M))$. Thus f is said to be almost continuous if it is almost continuous at each point x of X.

The concepts of almost continuous functions from one topological space into another which have been stated above are independent of each other. This can be shown by considering the following examples:

1. Let $\mathbb{R}$ represent the real numbers with the standard topology. Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be given by $f(x) = \begin{cases} -x & , \quad x \quad irrational \\ x & , \quad x \quad rational \end{cases}$ where f is an almost continuous function that is not continuous (Long and McGehee, 1970).
2. Let $\mathbb{R}$ represent the real numbers with the standard topology and define $f: \mathbb{R} \longrightarrow \mathbb{R}$ as $f(x) = \begin{cases} \sin \frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$, where f is an almost continuous function which is not continuous (Long and McGehee, 1970).
3. Let $\mathbb{R}$ represent the real numbers and $\mathcal{T}$ be that it consists of $\emptyset, \mathbb{R}$ and the complements of all countable subsets of $\mathbb{R}$. Let the set $X = \{a, b\}$ and let $\mathcal{T}^* = \{X, \emptyset, \{a\}\}$. Let $f: (\mathbb{R}, \mathcal{T}) \longrightarrow (\mathbb{R}, \mathcal{T}^*)$ be defined as follows: $f(x) = \begin{cases} a & , \quad x \quad rational \\ b & , \quad x \quad irrational \end{cases}$. Then f is an almost continuous function at each point of $\mathbb{R}$, but not continuous at $x \in \mathbb{R}$ if x is rational (Singal and Singal, 1968).

Example one above gives a function f which is almost continuous in the sense of Husain (1966) but not almost continuous in the sense of Stallings (1959). Furthermore such a function f is not a connected function, that is f does not preserve connected subsets of X and is not a connectivity function (the graph map $g(x) = (x, f(x))$ from X into $X \times Y$ is not a connected function). Example two gives a function that is almost continuous in the sense of Stallings (1959), is connected and is a connectivity function but is not almost continuous in the sense of Husain (1966). Example three above shows that an almost continuous function in the sense of Singal and Singal

(1968) need not be an almost continuous in the sense of Husain (1966). From example one above, clearly an almost continuous function in the sense of Husain (1966) need not to be almost continuous in the sense of Singal and Singal (1968). An example of a function which is almost continuous in the sense of Singal and Singal (1968) but not in the sense of Stallings (1959) was constructed by Herrington *et al.* (1974). The article considered X as the set of real numbers with the topology $\mathcal{T}$ consisting of the usual open sets together with the sets of the form $U \cap D$ where U was taken as an open set in the usual topology and D the set of all irrational numbers. The article supposed that $f: [0,1] \longrightarrow (X, \mathcal{T})$ be defined by $f(x) = x$. Then f satisfied to be almost continuous in the sense of Singal and Singal (1968) and also in the sense of Husain (1966). Since the only continuous functions on $[0,1] \longrightarrow (X, \mathcal{T})$ are the constant functions, f is not almost continuous in the sense of Stallings (1959). Almost continuous function may fail to be continuous as stated by Singal and Singal (1968). It considered $(\mathbb{R}, \mathcal{T})$ to be the space as in example three above and U to be usual topology for $\mathbb{R}$. Then an identity mapping $i: (\mathbb{R}, U) \longrightarrow (\mathbb{R}, \mathcal{T})$ is an almost continuous function but not continuous at any point.

Since these concepts of almost continuous functions are clearly independent of each other, this study has therefore considered the concept of almost continuous function as defined by Stallings (1959) while studying fixed point theorems for connectivity maps; in looking at the invariance of topological properties from cofinite spaces to the Euclidean $\mathbb{R}^n$ spaces, in defining *pseudoderivative* and finally in modeling cofinite topological manifolds. *Pseudoderivative* on almost continuous functions which is expected to facilitate the development of *pseudodifferentiable* manifolds on cofinite space is defined with almost continuous functions being approximated by a differentiable function for every neighborhood of zero. From this, it is important to focus on the concept of almost continuous functions whose graph can be approximated by the graph of the continuous function. Clearly it is the concept of almost continuous functions defined by Stallings (1959) where every open neighborhood of the graph of almost continuous functions contains the graph of some continuous functions.

2.3 Topological manifold

A topological space M is a topological manifold of dimension n if the following conditions hold:

1. M is a Hausdorff space, that is for any pair of distinct points $p, q \in M$, there exists open neighborhoods $p \in U$ and $q \in V$ such that $U \cap V = \emptyset$

2. M is locally Euclidean, that is for any $p \in M$ there exists a neighborhood U of p which is homeomorphic to an open subset $V \subset \mathbb{R}^n$.
3. M has a countable basis of open sets, that is M can be covered by countably many of such neighborhoods.

This has been shown by Lee (2000). A coordinate chart on M can be considered as a pair (U, φ) where $U \subset M$ is an open subset and $\varphi: U \longrightarrow \mathbb{R}^n$ is an injective function such that $\varphi(U)$ is open in $\mathbb{R}^n$. Since a manifold M is one of the models that one can do some analysis locally, the charts on it should have some compatibility. Two coordinate charts (U_1, φ_1) and (U_2, φ_2) on the manifold M are said to be compatible if $\varphi_i(U_1 \cap U_2)$ is open in $\mathbb{R}^n$ for some $i = 1,2$ and $\varphi_2 {\varphi_1}^{-1}: \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$. A collection of these charts say $B = \{(U_i, \varphi_i)\}$ of pairwise compatible charts with $\bigcup_i U_i = M$ is called atlas. A differentiable structure on M is an equivalence class of atlases (or equivalently it is a maximal atlas on M). Clearly two atlases are said to be compatible if their union is an atlas and since compatibility is an equivalence relation; every atlas is contained in a maximal one; the union of all atlases compatible with it. Therefore when a topological manifold is equipped with a differentiable structure, then the space becomes a differentiable manifold (Lee, 2000). The idea of topological manifold being equipped with a differentiable structure and becoming a differentiable manifold is necessary in this study especially when modeling is done on a non-Hausdorff space using almost continuous functions as maps. The bijection between the cofinite spaces shown by the existence of the inverse function theorem require the composition of almost continuous functions similar to the case of continuous bijection shown above. Therefore the knowledge of topological manifold is necessary as the study consider its modeling on the cofinite space.

2.4 Submanifolds

Under this, the concepts like Fréchet derivative, partition of unity, implicit function theorem and inverse function theorem are given consideration since they guarantee the existence of submanifolds, bringing in differentiability on manifolds. A subset N of a manifold M is known as a submanifold if for each $x \in N$, there is a chart (U, u) of M such that $u(U) \cap F_U$ where F_U is a closed linear subspace of the convenient model space E_U. Then clearly N is itself a manifold with $(U \cap N, u|_{U \cap N})$ as charts (Kriegl and Michor, 1997).

A continuous linear operator $L: N \longrightarrow M$ where N and M are Banach spaces is said to be Fréchet derivative of a function $f: N \longrightarrow M$ at a point $x \in N$ if $f(x+h) = f(x) + Lh + o(h)$ as $h \longrightarrow 0$. It is noted that $L = f\,'(x)$ is the Fréchet derivative so as to distinguish it from a different kind of derivative in normed spaces and it comprises of all terms that are linear in h (and possibly its derivatives). Higher order terms in h (and derivatives) comprises the remainder term $(o(h))$ (Griffel, 1985). The Fréchet derivative in infinite dimensional space is an extension of classical derivative in finite spaces. Most of the results of the classical derivatives can easily be generalized to Fréchet derivatives, for example the usual sum rule and product rule in case of functions of two or more variables apply to Fréchet derivatives. Other rules like chain rule and the implicit function theorem can as well be extended to Fréchet differentiable operators. This study is to define a derivative known as *pseudoderivative* by modifying the Fréchet derivative in the neighborhood of zero. The properties of this *pseudoderivative* are intended to be shown by the usual sum rule and product rule that have been known to be valid for classical derivatives and also extended to Fréchet derivatives.

A partition of unity on a differentiable manifold M is a collection $\{\psi_i\}_{i\in I}$ of smooth functions such that:

1. $\psi_i(p) \geq 0$ for every $p \in M$
2. the collection of supports $\{supp\ \psi_i : i \in I\}$ is locally finite
3. $\sum_{i\in I} \psi_i(p) = 1$ for every $p \in M$ (Borisovich *et al.*, 1985).

Partition of unity has also been used to model manifolds bringing in the creation of submanifolds which makes manifold to be useful since analysis and calculus can be done on it. This has been used in cases where modeling does not require a derivative.

Inverse function theorem defined by letting $P: U \subseteq F \longrightarrow V \subseteq G$ be a smooth map between Banach spaces. Suppose that for some $f_0 \in U$ the derivative $DP\left(f_0\right): F \longrightarrow G$ is an invertible linear map. Then we can find neighborhoods $\widetilde{U}$ of f_0 and $\tilde{V}$ of $g_0 = P(f_0)$ such that the map P gives a one to one map of $\widetilde{U}$ and $\tilde{V}$, and the inverse map $P^{-1}: \tilde{V} \subseteq G \longrightarrow \widetilde{U} \subseteq F$ is continuous (Hamilton, 1982). This inverse function theorem is important as one is able to know when a differentiable function can be inverted locally and whether the local inverse is also differentiable.

This will be useful when modeling using almost continuous functions as maps since it proves the bijection between the two cofinite spaces.

Implicit function theorem on the other hand was also defined by letting A, B and C be affine spaces modeled on the Banach spaces V, W and X. Suppose $U \subset A$ and $V \subset B$ are open, $F: U \times V \longrightarrow C$ is smooth, $(p_0, q_0) \in U \times V$ and $F(p_0, q_0) = C$. Assume that the second partial derivative, $(dF)^2_{(p_0,q_0)}: W \longrightarrow X$ is invertible. Then there exists a neighborhood $U' \subset U$ of p_0 and a smooth function $g: U' \rightarrow V$ such that $F(p, g(p)) = C$ for all $p \in U'$ and $g(p_0) = q_0$. Furthermore $dg_p = -(dF)^2_{(p,g(p))} \circ (dF)^1_{(p,g(p))}$. The knowledge of the implicit function theorem has been useful in modeling on spaces with Hausdorff properties especially where a derivative is needed for modeling.

The inverse function theorem and implicit function theorem are related. It had been shown by Lerman (2005) that the inverse function theorem implies the implicit function theorem in finite dimensional spaces. The article considered the case of a function $f: \mathrm{R}^2 \rightarrow \mathrm{R}$ as follows; Assume that there is a point $(a, b) \in \mathbb{R}^2$ so that $\frac{\partial f}{\partial y}(a, b) \neq 0$. Then there is a neighborhood U of point a in $\mathbb{R}$, V of b in $\mathbb{R}$ and a function $g: U \longrightarrow V$ so that $f(x, g(x)) = c$ for $x \in U$. Here $c = f(a, b)$.

Proof

Consider the map $H(x, y) = (x, f(x, y))$. It is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. We argue that it is invertible near the point (a, b). Indeed, its differential $DH(a, b)$ is $\begin{vmatrix} 1 & 0 \\ \frac{\delta f}{\delta x}(a,b) & \frac{\delta f}{\delta y}(a,b) \end{vmatrix}$. Hence det $DH(a, b) = 1 \cdot \frac{\partial f}{\partial y}(a, b)$, which is not zero by assumption. Hence $DH(a, b)$ is invertible. Hence by the inverse function theorem, the map H is invertible near (a, b). Denote the inverse by G. It is of the form $G(u, v) = (G_1(u, v), G_2(u, v))$ for some real - valued functions G_2, G_1 defined on a neighborhood of $H(a, b) = (a, f(a, b) = (a, c))$ in $\mathbb{R}^2$. Since G and H are inverses of each other, $(u, v) = H(G(u, v)) = H(G_1(u, v), G_2(u, v)) = (G_1(u, v), f(G_1(u, v), G_2(u, v)))$ for all (u, v) near (a, c). Therefore $u = G_1(u, v)$……….. (i) and $v = f(G_1(u, v), G_2(u, v))$……… (ii).

Substituting (i) into (ii), we get $v = f\big(u, G_2(u,v)\big)$ for all u near a, all v near c. Let $v = c$, $u = x$, we get $c = f\big(x, G_2(x,c)\big)$. Let $g(x) = G_2(x,c)$, we then have $c = f\big(x, g(x)\big)$.

It had been noted by Lerman (2005) that the above result showing that the inverse function theorem implying the implicit function theorem is also valid in infinite dimensional spaces. The cofinite spaces on which modeling are to be done is an infinite dimensional spaces hence the fact that inverse function theorem implies implicit function theorem holds for them. Therefore during modeling, the inverse function theorem will be used which intern guarantee the existence of implicit function theorem that usually holds for modeling on Hausdorff spaces. The implicit function theorem on T_2 – spaces has been used to get submanifolds thus using inverse function theorem in this study will lead to creation of submanifolds.

2.5 Euclidean space $\mathbb{R}^n$

The Euclidean space is the set of all n-tuples of real numbers $\mathbb{R}^n := \{(p_1, p_2, \dots p_n | p_i \in \mathbb{R})\}$. $p \in \mathbb{R}^n$ if and only if $p = (p_1, p_2, \dots, p_n)$. The Euclidean space $\mathbb{R}^n$ is a topological space since there is a standard notion of open sets and the properties of a topological space hold. By the fact that a point in a topological manifold has a neighborhood that is homeomorphic to some open subset in the n dimensional Euclidean $\mathbb{R}^n$ spaces, is enough to show that topological properties hold in the Euclidean spaces $\mathbb{R}^n$. The open sets in this space are the sets $U \subset \mathbb{R}^n$ such that every point in U lies in an open interval wholly contained in U, that is the point $a \in U$ implies that there exists $x, y \in \mathbb{R}^n$ such that $a \in (x, y) \subseteq U$. The open sets in the Euclidean space $\mathbb{R}^n$ can as well be described in terms of neighborhoods. For every $x, y \in \mathbb{R}^n$, the open balls of some radius ε about x and y gives the required neighborhoods, that is B_x is an open ball of radius $\frac{r}{2}$ centered at x and B_y an open ball of radius $\frac{r}{2}$ centered at y. Since this holds for every pair of distinct elements of $\mathbb{R}^n$, it follows that the Euclidean space $\mathbb{R}^n$ is a Hausdorff space. The existence of open sets in the Euclidean space $\mathbb{R}^n$ is important especially when investigating the invariance of topological properties to that space. Since some topological properties hold in cofinite spaces, this study therefore investigates the invariance of these topological properties from the cofinite space to the n dimensional Euclidean spaces $\mathbb{R}^n$ with respect to almost continuous functions.

2.6 T_1 - space and Hausdorff space (T_2 - space)

A topological space X is a T_1 – space if and only if for any pair of distinct points $a, b \in X$, each of these points belongs to an open set which does not contain the other. In other words, there exists open sets G and H such that $a \in G,\ b \notin G$ and $b \in H,\ a \notin H$. These open sets G and H are not necessarily disjoint. A topological space X is said to be a T_2 – space if and only if for each pair of distinct points $a, b \in X$ belongs respectively to disjoint open sets. In other words, there exists an open sets G and H such that $a \in G,\ b \in H$ and $G \cap H = \emptyset$ (Lipschutz, 1965). Every Hausdorff space is a T_1 – space. This can be shown by letting $(X, \mathcal{T})$ be a Hausdorff topological space and x be an element of X. Suppose that $y \in X \setminus \{x\}$, then $x \neq y$. So there must be neighborhoods U_1 of x and U_2 of y with $U_1 \cap U_2 = \emptyset$. In particular $x \notin U_2$. This shows that each element y in $X \setminus \{x\}$ has a neighborhood U_2 such that $U_2 \subseteq X \setminus \{x\}$. Clearly $X \setminus \{x\}$ is an open set. This means that $\{x\}$ is a closed set. As x was an arbitrary element of X, this shows that $(X, \mathcal{T})$ is a T_1 – space. In this study we consider a cofinite space that is a T_1 – space and the n dimensional Euclidean $\mathbb{R}^n$ space that is a T_2 – space, which by the above argument satisfies the conditions of a T_1 – space.

2.7 Summary of the literature review

Manifolds have been modeled on several spaces that have either T_1 or T_2 as separation properties. This has led to the development of non-Hausdorff manifolds and Hausdorff manifolds respectively. Homeomorphisms have been used from Hausdorff manifolds to the Euclidean $\mathbb{R}^n$ spaces. Invariance of topological properties has also been investigated with respect to continuous functions and almost continuous functions from one topological space (either a discrete or indiscrete) to another topological space. This study therefore intends to investigate the invariance of compactness and its other notions as topological properties from cofinite spaces which are infinite spaces to the Euclidean $\mathbb{R}^n$ spaces with respect to almost continuous functions. The study also seeks to define *pseudoderivative* on almost continuous functions which intern is intended to help in getting *pseudodifferentiable* cofinite manifolds.

CHAPTER THREE
METHODOLOGY

3.1 Working technique

The cofinite topology is a topology defined with open sets as complements of the finite subsets of the space (cofinite subsets of the space) along with the empty set. This gives the coarsest topology satisfying the T_1 – axiom hence T_1 – topology. This study has investigated the invariance of topological properties from the T_1 – space with cofinite topology (cofinite space) to the Euclidean $\mathbb{R}^n$ spaces with respect to almost continuous functions as defined by Stallings (1959). It has also defined a version of derivative called *pseudoderivative* on almost continuous functions. Such *pseudoderivative* has been obtained by modifying the Fréchet derivative $f(x+h) - f(x) = f\,'(x)h + r(h)$ where for every neighborhood of zero U_0, $r(h) \subset U_0$. Where $g(x)$ for every x is an almost continuous function, if there existed a continuous function $f(x)$ such that $g(x) - f(x) \subset U_0$ for every U_0, $g'(x)$ was taken as *pseudoderivative* on an almost continuous function $g(x)$ provided that $g(x+h) - g(x) = g'(x)h + r(h)$. Since a cofinite topology can be defined on a cofinite space, this study has developed a topological manifold on such a cofinite space using almost continuous functions defined by Stallings (1959) as maps. Culmination of modeling have been achieved by showing the existence of an inverse function theorem as a bijection between the cofinite spaces.

3.2 Some concepts used in developing results

The concepts outlined below are going to be useful in developing results in chapter four especially when investigating the invariance of topological properties from cofinite space to the Euclidean space $\mathbb{R}^n$ with respect to almost continuous functions.

3.2.1 Separable space

A subset A of a topological space X is said to be a dense subset in X if $\overline{A} = X$. This definition implies that a set A is dense if and only if every nonempty open set of X contains a point in A, meaning that any point in X can be approximated by points in A. A topological space X is said to be separable if it contains a countable dense subset. For instance a space $\mathbb{R}$ is separable since $\mathbb{Q}$ is countable and it is dense for every real number is a limit of rationals; for the same reason the Euclidean space $\mathbb{R}^n$ is separable (Munkres, 2000).

3.2.2 First and Second countable spaces

A space X is a first countable space if at each point p of the space, there is a countable local basis, that is a countable collection of open neighborhoods of $p \in X$ such that each open set containing p, contains a member of the collection. A topological space X is said to satisfy the second axiom of countability or to be second countable if it has a countable basis for its topology (Lipschutz, 1965). Clearly every second countable space is first countable for if $\mathfrak{B}$ is a countable basis for the topology of X, then the collection $\mathfrak{B}_x = \{B \in \mathfrak{B} | x \in B\}$ is a neighborhood basis for x and it is countable. The Euclidean space $\mathbb{R}^n$ is second countable because the collection $\mathfrak{B} = \{B_r(x) | x \in \mathbb{Q}^n, r > 0, r \in \mathbb{Q}\}$ consisting of open balls of rational radius around points with rational coordinates is a basis for the topology and $\mathfrak{B}$ is a countable collection. Suppose X is a second countable space; that is X has a countable basis, then a subset A of X consists of one element from each of the basis elements. Clearly A is countable and $\overline{A} = X$ showing that X is separable (Gichuki, 1996). From this, clearly the Euclidean space $\mathbb{R}^n$ being a second countable space, is also a separable space.

3.2.3 Compact space

A family $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of subsets of a space X is called a cover for X if $X = \bigcup_{\alpha \in A} U_\alpha$. It is called an open cover if each U_α is an open set. It is called finite if the index set A is finite. A sub-family $\mathfrak{A}$ of a cover $\mathcal{U}$ is called a subcover for X if $X = \bigcup \mathfrak{A} = \{\mathcal{U}_\alpha : U_\alpha \in \mathfrak{A}\}$. A topological space X is said to be a compact space if every open cover of X has a finite subcover. A set K in a space X is said to be compact in X if it is compact in the subspace topology (Deshpande, 1990). Equivalently a topological space X is compact if it satisfies the finite intersection property; that is if every family of closed subsets whose intersection is empty contains a finite subfamily whose intersection is empty. The concept of compactness is not nearly as natural as the concept of connectedness. However it limits the number of open sets in a topology since every open cover of a compact topological space must contain a finite subcover. From the definition of compactness every finite space is compact. Compact topological spaces exhibit many of the delightful properties of closed and bounded subsets of $\mathbb{R}$. This concept is elaborately defined since its invariance and its other notions are investigated from the cofinite space to the Euclidean space $\mathbb{R}^n$ with respect to almost continuous functions.

3.2.4 Sequential compactness

A topological space $(X, \mathcal{T})$ is a sequential space if and only if every sequentially open (closed) set is open (closed). Let X be a topological space. If (x_n) is a sequence of points of X and if $n_1 < n_2 < \cdots < n_i < \cdots$ is an increasing sequence of positive integers, then the sequence y_i defined by setting $y_i = x_{n_i}$ is called a subsequence of the sequence x_n (Munkres, 2000). Suppose X and Y are topological spaces and $f: X \longrightarrow Y$ is a map, then we say that f is sequentially continuous at a point $p \in X$ if for every sequence $\{x_i\}_{i=1}^{\infty}$ of elements in X which converges at $p \in X$, $\{f(x_i)\}_{i=1}^{\infty}$ converges to $f(p)$ as a sequence in Y. If f is continuous at p in the usual sense, then f is sequentially continuous at p. Conversely sequential continuity at p implies continuity at p. Clearly continuity of a function at a point p implies sequential continuity of the function at the same point and hence continuity of a function implies sequential continuity. Thus the space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence (Joshi, 1983). In the Euclidean space $\mathbb{R}^n$, every bounded sequence has a convergent subsequence. Clearly suppose (x_n) is a bounded sequence in $\mathbb{R}^n$, then it is contained in some closed and bounded cube M and by Heine - Borel theorem, it is compact. This shows that M is sequentially compact and implies that (x_n) has a convergent subsequence.

3.2.5 Heine - Borel theorem

Let $A = [a, b]$ be a closed and bounded interval and let $\mathcal{G} = \{G_i : i \in \mathcal{I}\}$ be a class of open intervals which cover A, that is $A \subset \bigcup_i G_i$. Then $\mathcal{G}$ contains a finite subclass, say $\{G_{i_1}, \dots, G_{i_m}\}$ which also covers A, that is $A \subset G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_m}$ (Lipschutz, 1965). This theorem is important in this research as it supports the existence of other notions of compactness in the Euclidean space $\mathbb{R}^n$. This will enable the investigation of the invariance of these other forms of compactness from the cofinite space to the Euclidean space $\mathbb{R}^n$ with respect to almost continuous functions.

3.2.6 Bolzano - Weierstrass theorem

In a compact space S, a subset A containing an infinite collection of points possesses a limit point (Cairns, 1961). For the case of Euclidean space $\mathbb{R}^n$, every bounded infinite subset of $\mathbb{R}^n$ has at least one accumulation point in $\mathbb{R}^n$. For if B is a bounded infinite subset of $\mathbb{R}^n$, it is contained in some closed cube $I_n = [-n \times n] \times [-n \times n] \times \dots \times [-n \times n]$. Since I_n is closed and bounded, it is compact by Heine - Borel theorem and since B is an infinite subset of a compact set I_n, it must have an accumulation point in $\mathbb{R}^n$. This theorem too supports the existence of other notions of compactness in the Euclidean space $\mathbb{R}^n$. Therefore this study investigates their invariance from the cofinite space to the Euclidean space $\mathbb{R}^n$ with respect to almost continuous functions.

3.2.7 Pseudocompactness

A topological space X is said to be pseudocompact if every continuous real valued function on X is bounded. On the other hand, a topological space X is said to be countably compact if every countable open cover of X has a finite subcover. Every compact space X is countably compact for if A is a countable cover of a compact space X, then A has a finite subcover since all open covers of X have finite subcovers. This implies that X is countably compact, though the converse of this statement may not hold. From the two definitions above, every countably compact space X is pseudocompact although the converse does not hold. This is true since for a continuous function f on X, the set $S_n = \{x: |f(x)| < n\}$ forms a countable cover of X whose finite subcover yields a bound for the absolute value of f (Steen and Seebach, 1970). This shows that when a space X is compact, then it is countably compact and hence pseudocompact. This statement therefore holds for the cofinite space which is compact and thus pseudocompact.

3.2.8 Limit point compactness

A point x is a limit point of a subset A of a topological space X if and only if every neighborhood U_x of x contains a point of A other than x (Kelly, 1975). A topological space X is said to be limit point compact if every infinite subset of X, $A \subset X$ has a limit point (Munkres, 2000). One would therefore say that a space is limit point compact if it contains no infinite closed discrete subspaces. A topological space is said to be weakly countably compact if every infinite set has a limit point. A space that is limit point compact is also known as weakly countably compact. In a T_1 – space weak countable compactness is equivalent to countable compactness (Steen and Seebach, 1970). Hence in T_1 – spaces countable compactness is equivalent to limit point compactness. Since cofinite space is a T_1 – space which is compact, then it is also limit point compact.

3.3 Some of the results developed on cofinite spaces

These results that have been developed on cofinite spaces are important as they show some of the properties possessed by cofinite spaces. This will enable the investigation of invariance of these properties from the cofinite spaces to the Euclidean space $\mathbb{R}^n$ with respect to almost continuous functions.

3.3.1 Theorem

Let X be an infinite set with finite complement topology. Then any infinite subset A of X is dense in X (Deshpande, 1990).

Proof

Since X is an infinite set, by definition we know that the closed subset of X are the finite subsets of X and X itself. Specifically the only infinite closed subset of X is X itself. Let A be an infinite subset of X, then $A \subseteq \overline{A}$. Hence the closed set $\overline{A}$ must be infinite; the only infinite closed subset is X and therefore $\overline{A} = X$. Hence A is dense in X thus making X be a separable space. ∎

3.3.2 Theorem

Any space X with cofinite topology is compact (Deshpande, 1990).

Proof

Let X be an infinite space with cofinite topology. If V is any open cover for X, then one of the members of V, say V_0 covers all but finitely many points of X, since $V_0 = X \setminus \{x_1, \dots, x_n\}$ for some $x_1, x_2, \dots, x_n$ in X. There is some $V_j \in V$ covering each x_j, so that $\{V_0, V_1, \dots, V_n\}$ forms a finite subcover for X. Thus the cofinite space X is compact. ∎

3.3.3 Theorem

Every cofinite space is sequentially compact (Deshpande, 1990).

Proof

Suppose (x_n) is any infinite sequence in X and U is any neighborhood of a point p; say $U = X \setminus \{p_1, p_2, \dots, p_n\}$ for some p_j's in X. Since $\{x_n\}$ is an infinite set, there exists n_0 such that for $n \geq n_0$, $x_n \neq p_j$ (where $j = 1, 2, \dots, n$) that is, $x_n \in U$ for $n \geq n_0$. In other words, every infinite sequence converges to every point p in X. If (x_n) is such that the set $\{x_n\}$ is finite, then (x_n) certainly has a convergent subsequence. Thus X is sequentially compact. ∎

3.3.4 Theorem

Compactness implies limit point compactness but not conversely (Munkres, 2000).

Proof

Let X be a compact space. Given a subset A of X, we wish to show that if A is infinite, then A has a limit point. We show the contrapositive that if A has no limit point, then A must be finite. So suppose A has no limit point, then A contains all its limit points so that A is closed. Furthermore for each $a \in A$ we can choose a neighborhood U_a of a so that U_a intersects A in the point a alone. The space X is covered by open set $X \setminus A$ and the open sets U_a. The space X being compact, it can be covered by finitely many of these sets. Since $X \setminus A$ does not intersect A and each set U_a contains only one point of A, the set A must be finite. ∎

3.3.5 Theorem

Any infinite set $Z \subset X$ of a compact space X has a limit point in X (Borisovich *et al.*, 1985).

Proof

Assume a contrary that $Z' = \emptyset$, then $\overline{Z} = Z$. Then Z is closed and consequently compact. On the other hand, each point $\mathfrak{a} \in Z$ is isolated in X. This implies that there exists an open neighborhood $\Omega(\mathfrak{a})$ in X such that $\Omega(\mathfrak{a}) \cap Z = \mathfrak{a}$. Neighborhoods $U(\mathfrak{a}) = \Omega(\mathfrak{a}) \cap Z$ that are open in Z forms an infinite covering of the space Z from which a finite subcovering cannot be selected and so we arrive at a contradiction to the assumption that Z is compact. ∎

3.3.6 Theorem

Cofinite space is limit point compact (Adams and Franzosa, 2008).

Proof

Let X be an infinite set with finite complement topology. We show that X is limit point compact. Thus let B be an infinite subset of X. We claim that every point of X is a limit point of B. If $x \in X$ and U a neighborhood of x, then since U contains all but finitely many points of X, it intersects B in infinitely many points. In particular, U intersects B in points other than x, implying that x is a limit point of B. Therefore every infinite subset B of X has a limit point, implying that X is limit point compact. ∎

CHAPTER FOUR

RESULTS AND DISCUSSION

This chapter defines manifolds based on cofinite spaces using almost continuous functions as maps. This is done in two parts: the first part looks at the invariance of topological properties from a T_1 – space with cofinite topology (cofinite space) with respect to almost continuous functions to the Euclidean $\mathbb{R}^n$ spaces, thereafter *pseudoderivative* is defined on almost continuous functions. The second part involves modeling of a cofinite topological manifold $\mathcal{W}$ using *pseudodifferentiable* almost continuous functions bringing in the inverse function theorem which have been shown to imply the implicit function theorem in infinite cofinite spaces.

4.1 Almost continuous function on a cofinite space

The study defines an almost continuous function in every neighborhood of zero and shows a result of an almost continuous function as a map between any two cofinite spaces.

4.1.1 Definition

A function g is an almost continuous function at x_0 if there exists a continuous function f such that $(f - g)(x_0) \subset U_0$ for every neighborhood of zero U_0.

4.1.2 Theorem

Let $X_1 = (X, \mathcal{T})$ be an infinite space with finite complement topology and let g be an almost continuous function. Then $X_2 = g(X_1)$ is an infinite space with finite complement topology.

Proof

Given that X_1 is an infinite set with finite complement topology and if $U \subset X_1$, then U' consists of finite elements. If f is a continuous function from $X_1 \longrightarrow X_2$, then $f(U') \subset X_2$ consists of finite elements since f is a one to one map. Note that g is an almost continuous functions if for every neighborhood of zero $U_0 \subset X_2$, there exists a neighborhood $V \subset X_1$ such that $(f - g)(V) \subset U_0 \ \forall \ V \subset X_1$. Since $f - g$ is one to one, then U_0 consists of infinite elements and $(f - g)(V')$ must consist of finite elements since V consists of infinite elements and V' consists of finite elements. This is due to the fact that X_1 is a cofinite space. ■

4.2 Invariance of topological properties

At this point, this study narrows down to one of the T_1 – spaces known as a cofinite space C, that is an infinite set X having the cofinite topology. Attention is paid to separability as a cardinality property, compactness and its other notions such as sequential compactness, limit point compactness and pseudocompactness. A property is said to be a topological invariant (or topological property) if whenever one space possesses a given property, any space homeomorphic to it also possesses the same property. The properties of topological spaces that remain unchanged when space X is mapped onto a space Y by means of a function are said to be invariant of that function. Thus if a property P is known to be invariant of the function f, it can be determined whether a topological space Y has the property P by only showing that $Y = f(X)$ for some function f and some topological space X having property P. The behaviours of these topological properties are studied with respect to almost continuous functions from the cofinite space C to the Euclidean $\mathbb{R}^n$ spaces. These are done with respect to continuous functions and then the remark 4.2.1 noted below is used to make a conclusion.

4.2.1 Remark

It has been shown in the literature review that if a property P is invariant of continuous functions, it must be invariant of continuous bijections and hence invariant of almost continuous bijections in the class of T_1 – spaces. Those topological properties which are invariant of continuous functions are also invariant of almost continuous functions in the class of T_1 – spaces.

4.2.2 Separability

Separability is obtained from the concept 3.2.1 and from 3.2.2; it is concluded that every space that satisfies the second axiom of countability is separable. Since the Euclidean $\mathbb{R}^n$ space is a second countable space as shown in 2.5, it is a separable space. From theorem 3.3.1, clearly a cofinite space C is a separable space.

4.2.3 Theorem

Let f be a continuous function from a separable cofinite space C to the Euclidean space $\mathbb{R}^n$. If A is a countable dense subset of C, then $f(A)$ is a countable dense subset of $f(C)$ in the Euclidean space $\mathbb{R}^n$.

Proof

Consider a continuous function $f: C \longrightarrow \mathbb{R}^n$ where C is a separable cofinite space. Since A is a countable dense subset of C, that is $\overline{A} = C$, it is then shown that $f(\overline{A})$ is dense in $f(C)$ which is a subset of the Euclidean $\mathbb{R}^n$ space due to the continuity of f. Now for $f: X \longrightarrow Y$ where X and Y are topological spaces being continuous and $A \subseteq X$, letting $x \in \overline{A}$ implying that $f(x) \in f(\overline{A})$. We let V be a neighborhood of $f(x)$. By continuity of f, $f^{-1}(V)$ is an open set in X containing x. Thus we have $f^{-1}(V) \cap A \neq \emptyset$ implying that $f(x) \in \overline{f(A)}$. Therefore $f(\overline{A}) \subseteq \overline{f(A)}$. From this, X can be taken as a separable cofinite space C and Y be taken as the Euclidean space $\mathbb{R}^n$. But $A \subseteq \overline{A}$ and $\overline{A} \subseteq C$ by which would imply that $f(A) \subseteq f(\overline{A}) \subseteq \overline{f(A)} \subseteq f(C)$. Clearly $f(\overline{A}) \subseteq f(C)$ and this would mean that $f(\overline{A}) = f(C)$. That is $f(C)$ has a countable dense subset $f(\overline{A})$, showing that $f(C)$ is separable as a subset of $\mathbb{R}^n$. Therefore separability is invariant of continuous functions from cofinite space C to Euclidean $\mathbb{R}^n$ space. From remark 4.2.1; separability is invariant of almost continuous functions from cofinite space C to Euclidean $\mathbb{R}^n$ space. ■

4.2.4 Compactness

The concept of compactness was no doubt motivated by the property of closed and bounded interval as outlined in the concept 3.2.3, which also describe a compact space. From theorem 3.3.2, it is obtained that every infinite space X with cofinite topology is compact. Suppose $(\mathbb{R}^n, \mathcal{T}_{cof})$ is a Euclidean space with cofinite topology. Then the open sets are any sets with a complement that is finite. Clearly every open set is infinitely large and contains all but finitely many points of $\mathbb{R}^n$. The closed sets of $\mathbb{R}^n$ consists of finitely many points, so a finite subcover can always be found by covering each point one at a time with the open set whose finite complement is each of the other points in that closed set. The remaining sets in $\mathbb{R}^n$, the open sets, clearly have a finite subcover of open sets, namely themselves. Therefore any subset of Euclidean space $\mathbb{R}^n$ with the finite complement topology is compact.

4.2.5 Theorem

Suppose C is a compact cofinite space and $f: C \longrightarrow \mathbb{R}^n$ is a continuous function, then $f(C)$ is a compact subset of the Euclidean $\mathbb{R}^n$ space.

Proof

Let $\mathcal{A} = \{U_\alpha : \alpha \in \zeta\}$ be any open cover for $f(C)$. Then one of the members of U_α say U_β covers all but finitely many points of $f(C)$ since $U_\beta = f(C) \setminus \{a_1, a_2, \dots, a_m\}$ for some $a_1, a_2, \dots, a_m$ in $f(C)$. Therefore $f(C) = \bigcup_{\alpha\in\zeta} U_\alpha$

$$C = f^{-1}\left(\bigcup_{\alpha\in\zeta} U_\alpha\right) = \bigcup_{\alpha\in\zeta} f^{-1}(U_\alpha)$$

Since f is continuous, $f^{-1}(U_\alpha)$ is open in C for some $\alpha \in \zeta$. Then $\{f^{-1}(U_\alpha) : \alpha \in \zeta\}$ is an open cover for C. Because C is compact, there exists $\alpha_1, \alpha_2, \dots, \alpha_m \in \zeta$ such that $C = f^{-1}(U_\beta) \cup \left[\bigcup_{i=1}^{m} f^{-1}(U_{\alpha_i})\right] = f^{-1}(U_\beta) \cup f^{-1}\left[\bigcup_{i=1}^{m} U_{\alpha_i}\right] = f^{-1}\left[U_\beta \cup \left(\bigcup_{i=1}^{m} U_{\alpha_i}\right)\right]$

$f(C) = U_\beta \cup \left(\bigcup_{i=1}^{m} U_{\alpha_i}\right)$. Hence $\{U_\beta \cup U_{\alpha_i} : i = 1, \dots, m\}$ is a finite subcover for $\mathcal{A}$. From this, compactness is invariant of continuous functions from a compact cofinite space C to the Euclidean $\mathbb{R}^n$ space. From remark 4.2.1; compactness is invariant of almost continuous functions from the cofinite space C to the Euclidean $\mathbb{R}^n$ space. ■

Attention is now paid to other notions of compactness which are equivalent to compactness in metric spaces but are nonequivalent topological properties in arbitrary topological spaces.

4.2.6 Sequential compactness

As the name suggests, sequential compactness is a sequence version of compactness. The concept 3.2.4 gives a sequentially compact space and from the theorem 3.3.3, it is evident that the cofinite space X is sequentially compact. Suppose $\mathbb{R}^n$ has the Euclidean topology, then an example of a compact set in $\mathbb{R}^1$ would be $[0,1]$. This can be seen from the definition of a compact set and that this interval has a finite subcover. For $(\mathbb{R}^n, \mathcal{T}_{Eucl})$, a theorem that defines all its compact subsets can be obtained, that is a subset A of a real Euclidean space $\mathbb{R}^n$, $A \subseteq (\mathbb{R}^n, \mathcal{T}_{Eucl})$ is compact if and only if A is closed and bounded. This is a generalization of Heine - Borel theorem and by Bolzano - Weierstrass theorem; any closed and bounded subset of the Euclidean space $\mathbb{R}^n$ is sequentially compact. Clearly if B is a closed and bounded subset of the Euclidean $\mathbb{R}^n$ space, then B is sequentially compact.

4.2.7 Theorem

Let f be a continuous function from a sequentially compact cofinite space C to a closed and bounded subset B of the Euclidean $\mathbb{R}^n$ space. Then $f(C)$ is a sequentially compact subset of the Euclidean space $\mathbb{R}^n$.

Proof

Consider a continuous function $f: C \longrightarrow B$ where C is a sequentially compact cofinite space and B is a closed and bounded subset of the Euclidean $\mathbb{R}^n$ space. Because of continuity of f, $f(C) \subset B$. Let $(y_1, y_2, \dots)$ be a sequence in $f(C)$. Then there exists $x_1, x_2, \dots \in C$ such that $f(x_n) = y_n$ for every $n \in \mathbb{N}$. But C is sequentially compact, so the sequence $(x_1, x_2, \dots)$ contains a subsequence $(x_{i_1}, x_{i_2}, \dots)$ which converges to a point $p \in C$. Now f is continuous and hence sequentially continuous, so $\{f(x_{i_1}), f(x_{i_2}), \dots\} = \{y_{i_1}, y_{i_2}, \dots\}$ converges to $f(p) \in f(C)$. Thus $f(C)$ is sequentially compact with a sequence $(y_1, y_2, \dots)$ having a convergent subsequence $(y_{i_1}, y_{i_2}, \dots)$. Therefore sequential compactness is invariant of continuous functions from a cofinite space C to Euclidean $\mathbb{R}^n$ space. From remark 4.2.1; Sequential compactness is invariant of almost continuous functions from a cofinite space C to the Euclidean space $\mathbb{R}^n$. ■

4.2.8 Pseudocompactness

The concept outlined in 3.2.7 defines a pseudocompact space and by the fact that a compact space is pseudocompact, it is shown that a cofinite space C is also pseudocompact. A cofinite space C is a compact space as shown in theorem 3.3.2. The image of a compact space under any continuous function is compact. By Heine - Borel theorem, the compact subsets of the Euclidean space $\mathbb{R}^n$ are precisely the closed and bounded subsets. Hence a cofinite space is pseudocompact as its image under any continuous function to $\mathbb{R}^n$ is compact.

4.2.9 Theorem

Let f be a continuous function from a pseudocompact cofinite space C to the Euclidean space $\mathbb{R}^n$. Then $f(C)$ is a pseudocompact subset of the Euclidean space $\mathbb{R}^n$.

Proof

To show the continuity invariance of pseudocompactness to the Euclidean $\mathbb{R}^n$ space, a continuous function $f: C \longrightarrow \mathbb{R}^n$ is considered. Clearly $f(C)$ is a compact subset of $\mathbb{R}^n$ by continuity of f. By the definition in 3.2.5, $f(C)$ is bounded, that is f is a bounded function and $f(C)$ is closed since it is a compact subset of $\mathbb{R}^n$. This implies that the continuous function f attains its bounds as the supremum and infimum of $f(C)$ which are either in $f(C)$ or are the limit points. Since $f(C)$ is the continuous image of pseudocompact space C, it follows that $f(C)$ is pseudocompact. Hence pseudocompactness is invariant of continuous functions from the pseudocompact cofinite space C

to the Euclidean $\mathbb{R}^n$ space. From the remark 4.2.1; pseudocompactness is invariant of almost continuous functions from the cofinite space C to the Euclidean $\mathbb{R}^n$ space. ■

4.2.10 Limit point compactness

The concept in 3.2.8 gives a limit point compact space and from the theorems 3.3.5 and 3.3.6, it can be seen that the cofinite space C is a limit point compact space, that is any infinite subset $A \subset C$ contains limit point in C.

4.2.11 Theorem

Let C be a limit point compact cofinite space and f be a continuous function from C to the Euclidean space $\mathbb{R}^n$, then $f(C)$ is a limit point compact space in the Euclidean space $\mathbb{R}^n$.

Proof

Suppose K is a closed and bounded subset of $\mathbb{R}^n$, then it is compact since the closed and bounded subsets of the Euclidean $\mathbb{R}^n$ space are compact. If B is an infinite subset of K, then B is also bounded and by Bolzano - Weierstrass theorem, B has a limit point p. Since K is closed, the limit point p of B belongs to K, that is K is limit point compact. A continuous function $f: C \longrightarrow K$ is considered and because of continuity of f, $f(C) \subset K$. Since C contains an infinite set A whose limit point is in C, then $f(C) \subset K$ contains an infinite set $f(A)$ whose limit point is in $f(C)$. But $f(C)$ is a subset of a closed and bounded set $K \subset \mathbb{R}^n$ which is also limit point compact. Clearly the cofinite space that is limit point compact is continuous invariant to the Euclidean $\mathbb{R}^n$ space. Therefore limit point compactness is continuity invariant from cofinite space C to the Euclidean $\mathbb{R}^n$ space. Hence from remark 4.2.1; limit point compactness is invariant with respect to almost continuous function from cofinite space C to the Euclidean $\mathbb{R}^n$ space. ■

4.3 Topological manifold based on cofinite spaces

A topological space $\mathcal{W}$ is a topological manifold on a cofinite space if it has a countable base of open sets and for every neighborhoods $U(x_0)$ and $V(x_0)$ of a point $x_0 \in \mathcal{W}$ there exists almost continuous functions ϕ and ψ such that:

1. $\phi(U(x_0))$ and $\psi(V(x_0))$ maps $\mathcal{W}$ to cofinite spaces.
2. $\exists$ a bijective almost continuous function $h: \phi(U(x_0) \cap V(x_0)) \longrightarrow \psi(U(x_0) \cap V(x_0))$.

4.4 *Pseudoderivative* on almost continuous functions

A linear operator T is $D_{\beta,\gamma}$ *pseudoderivative* on an almost continuous function g at x_0 if there exists a $D_{\beta,\gamma}$ differentiable function f such that $(T - f')(x_0) \subset U_0$ for every neighborhood of zero, U_0.

4.4.1 Properties of *pseudoderivative* on almost continuous functions

Given that g is an almost continuous function and f is a continuous function both defined in an open neighborhood U_0 of a point x, then the *pseudoderivative* on almost continuous functions satisfies the following properties:

Linearity which consists of two parts:

a) $[\lambda(g - f)](x + h) - [\lambda(g - f)](x) = \lambda g(x + h) - \lambda g(x) - \lambda f(x + h) + \lambda f(x)$

$$= \lambda\big(g'(x) - f'(x)\big) \subset U_0$$

$$\therefore \big[\lambda\big(g(x) - f(x)\big)\big]' = \lambda\big(g(x) - f(x)\big)' \subset U_0$$

From the above, *pseudoderivative* of a constant times an almost continuous function can as well be given by $(\lambda g)'(x) = \lambda g'(x)$

b) Given two almost continuous functions g_1 and g_2 and a continuous function f we have the sum of *pseudoderivatives* given as

$$[(g_1 + g_2) - f](x + h) - [(g_1 + g_2) - f](x)$$

$$= g_1(x + h) + g_2(x + h) - f(x + h) - g_1(x) - g_2(x) + f(x)$$

$$= g_1(x + h) - g_1(x) + g_2(x + h) - g_2(x) - \big(f(x + h) - f(x)\big)$$

$$= \big[\big(g_1'(x) + g_2'(x)\big) - f'(x)\big] \subset U_0$$

$$\therefore (g_1 + g_2)'(x) = g_1'(x) + g_2'(x)$$

The product of *pseudoderivatives* on almost continuous functions can as well be shown by considering two almost continuous functions g and h and two differentiable functions f_1 and f_2 as follows:

$$[gh - f_1f_2](x + \alpha) - [gh - f_1f_2](x)$$

$$= gh(x + \alpha) - g(x)h(x + \alpha) + g(x)h(x + \alpha) - g(x)h(x) - f_1f_2(x + \alpha) + f_1(x)f_2(x + \alpha) - f_1(x)f_2(x + \alpha) + f_1(x)f_2(x)$$

$$= h(x+\alpha)[g(x+\alpha)-g(x)] - f_2(x+\alpha)[f_1(x+\alpha)-f_1(x)] + g(x)[h(x+\alpha)-h(x)] - f_1(x)[f_2(x+\alpha)-f_2(x)]$$

$$= [h(x+\alpha)g'(x) - f_2(x+\alpha)f_1'(x)] + [g(x)h'(x) - f_1(x)f_2'(x)] \subset U_0$$

From above such a product can be indicated as $(gh)'(x) = g(x)h'(x) + g'(x)h(x)$.

4.5 Topological manifold $\mathcal{W}$ on cofinite spaces

4.5.1 Definition

A piecewise linear map is a map composed of some number of linear segments defined over an equal number of neighborhoods.

4.5.2 Theorem

Given any compact space $P \subset X$ and any $f \in C[P]$, there exists a map $\Phi_\varepsilon : X \longrightarrow Y$ such that $f(x) - \Phi_\varepsilon(x) \subset V_0\ \forall x \in X$ where V_0 is a neighborhood of zero in Y…………...………...… (**)

Proof

Since f is uniformly continuous on the compact set P and given any neighborhood of zero $V_0 \subset Y$, there exists a neighborhood of zero $U_0 \subset X$ such that, for any $x, x' \in U_0$ with $x - x' \subset U_0$, we have $f(x) - f(x') \subset V_0$. Now pick $P_n : n \in \mathbb{N}$ such that $\bigcup P_n = P$ and let $x_m \coloneqq P_n$ where $m = 0, 1, \dots, n$. Define Φ_ε as follows: $\Phi(e) \coloneqq f(e)\quad \forall e \in P_1$ and $\phi_i(x) \coloneqq f(x_i)\ \forall x \in P_i, 1 \le i \le n$ so that, the map Φ_i is constantly equal to the value of f at P_i hence $f(x) - \Phi_\varepsilon(x) \subset V_0$ and $f(x) - f(x_i) \subset V_0$ since $x - x_i \subset U_0$. Therefore, (**) is satisfied on P. ■

4.5.3 Theorem

Let X and Y be cofinite linear topological spaces without norm and $f \in C[X]$, then there exists an almost continuous function $\Psi_\varepsilon : X \longrightarrow Y$ such that $f(x) - \Psi_\varepsilon(x) \subset V_0\ \forall\, x \in X$ where V_0 is a neighborhood of zero in Y.

This follows from the fact that almost continuous functions are subsets of piecewise linear map and continuous functions are subsets of almost continuous functions. The result of this theorem follows from theorem 4.5.2 above.

4.5.4 Manifold on a Hausdorff space

Developing a manifold on a T_2 – space can be achieved by applying the Urysohn's lemma on such a T_2 – space. This leads to the existence of a bicontinuous bijective function. When implicit function theorem or partition of unity is applied on the homeomorphism, we create a manifold on a Hausdorff space. In the same spirit, modeling of a manifold on a T_1 – space with cofinite topology requires similar steps. Almost continuous functions had been used by Gichuki (1996) in the class of T_1 – spaces. Since the cofinite spaces are among the class of T_1 – spaces, almost continuous functions also exist for them. The inverse function theorem is then applied to model a manifold on the cofinite topology existing on the T_1 – space. In this study, it has been shown that almost continuous functions map cofinite space to cofinite space. This almost continuous function has a version of derivative called a *pseudoderivative* which has almost all properties as the derivative in calculus that can be used to state and prove the inverse function theorem. The inverse function theorem in turn guarantees the existence of submanifolds in the manifolds based on cofinite spaces.

From the literature review, it is clear that the inverse function theorem implies the implicit function theorem in both finite and infinite dimensional spaces. Hence the inverse function theorem is valid for cofinite spaces which are infinite dimensional spaces.

4.6 Differentiable manifold on cofinite space

A topological space $\mathcal{W}$ is a differentiable manifold on a cofinite space if it has a countable base of open sets and for every neighborhoods $U(x_0)$ and $V(x_0)$ of a point $x_0 \in \mathcal{W}$ there exists *pseudodifferentiable* almost continuous functions ϕ and ψ such that: $\phi(U(x_0))$ and $\psi(V(x_0))$ maps $\mathcal{W}$ to cofinite spaces and there exists a bijective *pseudodifferentiable* almost continuous function $h: \phi(U(x_0) \cap V(x_0)) \longrightarrow \psi(U(x_0) \cap V(x_0))$.

4.6.1 Definition

Let P be an open set in $\phi(U(x_0))$. The map $f: P \longrightarrow \psi(V(x_0))$ is $D_{\beta,\gamma}$ differentiable at $x_0 \in P$, if there exists a linear continuous operator $f\,'(x_0): \phi(U(x_0)) \longrightarrow \psi(V(x_0))$, such that $R(x,h) \equiv f(x+h) - f(x) - f\,'(x_0)h$ satisfies the condition:

$\exists C \; \forall \; B \; \exists \; U : (h \in B + U, ah \in U, \; x - x_0 \in U) \implies R(x, ah) \in aC$ ……………………….. (i)

4.6.2 Theorem

Let P be an open set in $\phi(U(x_0))$, $f: P \longrightarrow \psi(V(x_0))$ is strictly $D_{\beta_{\mathcal{C}},\gamma_{\mathcal{C}}}$ differentiable at $x_o \in P$ and $f\,'(x_0)$ a linear homeomorphism of $\phi(U(x_0))$ onto the subspace (respectively onto a closed subspace) $\phi(U(x_0) \cap V(x_0)) \subset \psi(V(x_0))$, with induced topology. Then there exists such an open neighborhood N of x_0, such that $f|_N: N \longrightarrow \psi(V(x_0))$ is injective.

Proof

A case of $D_{\beta_{\mathcal{C}},\gamma_{\mathcal{C}}}$ differentiability is looked at. Without reducing generality, assume that $x_0 = 0$, $y_0 = 0$, $\phi(x_0) = \phi(U(x_0) \cap V(x_0))$ and $f\,'(x_0)$ is identity operator. On the contrary we look at the map $F: \phi(U(x_0) \cap V(x_0)) \longrightarrow \phi(U(x_0) \cap V(x_0))$, defined by the formula $F(y) = f\,'(x_0 + [f\,'(x_0)]^{-1}y) - f(x_0)$ for $y \in f^{-1}(x_0)(P - x_0)$.

Note that a linear homeomorphism takes a bounded set to a bounded set and a compact to a compact. In view of the above, f can be expressed as $f(x+h) - f(x) = h + R(x,h)$ where R satisfies (i) for $x_0 = 0$, $\gamma = \gamma_{\mathcal{C}}$ and $\beta = \beta_{\mathcal{C}}$. For $B \supset (2C) \cap \phi(x_0)$, where C is taken from (i), we find such a U, that $(h \in B + 2U, ah \in 2U, x \in U) \Longrightarrow R(x, ah) \in aC$ ……….……………... (ii)

We show that $f|_U: U \longrightarrow \psi(x_0)$ is injective. Let x, $x + h \in U$ and $f(x+h) = f(x)$. Then $h \in U - x \subset 2U$ and from (ii) follows that $R(x,h) \in C$.

Further $h + R(x,h) = f(x+h) - f(x) = 0$ i.e. $h = -R(x,h) \in C$. Let $\mu > 0$. Then since C is closed, $h \in (\mu C) \cap \phi(x_0) = \frac{1}{2}\mu[(2C) \cap \phi(x_0)] \subset \frac{1}{2}\mu B$. In this regard from (ii) follows that $-h = R(x,h) = R\left(x, \frac{1}{2}\mu(2\mu^{-1}h)\right) \in \frac{1}{2}\mu C$, since $2\mu^{-1}h \in B \subset B + 2U$.

If $K \subset \psi(x_0)$, $y \in \psi(x_0)$, $P_K(y) = \begin{cases} \infty, & y \notin \mu C (\forall\, \mu > 0) \\ \inf\{\mu > 0 : y \in \mu K\} \end{cases}$

Then it implies that $\mu = P_c(h) \leq \frac{1}{2}\mu$ and this is impossible since $\mu > 0$. Therefore $P_c(h) = 0$ and hence $h = 0$ in view of boundedness of C. From this instead of $\mathcal{N}$ in the theorem, U can be used. The closure of $\phi(x_0) = \phi(U(x_0) \cap V(x_0))$ in the case of D_{β_c,γ_c} differentiability of f was used in the choice of $B \supset (2C) \cap \phi(x_0)$, since $2C \cap \phi(x_0)$ is compact in $\phi(x_0)$, if C is compact in $\psi(x_0)$.

■

4.6.3 Theorem

Assume conditions of the theorem 4.6.2 are fulfilled (except closure of $\phi(U(x_0) \cap V(x_0))$ where R satisfies the condition: $\exists C\ \forall\ C'\ \exists V$:

$(h \in (C' + V) \cap \phi(U(x_0)), ah \in V \cap \phi(U(x_0)), x - x_0 \in V \cap \phi(U(x_0)) \Longrightarrow R(x, ah) \in aC$ (iii). (Here $\phi(U(x_0)) = \phi(U(x_0) \cap V(x_0))$, $f'(x_0) = \text{id}$). Then there exists such an open neighborhood H of x_0 such that $f|_H: H \longrightarrow f(H)$ is bijective and uniformly continuous together with the inverse map.

Proof

Under conditions of the proof of theorem 4.6.2, for $C' \supset 2C$ where C is taken from (iii) we look for such a V that the following condition is fulfilled:

$(h \in (C' + V) \cap \phi(x_0), ah \in (2V) \cap \phi(x_0), x \in V \cap \phi(x_0)) \Longrightarrow R(x, ah) \in aC$ ………..…... (iv)

Choose an arbitrary V'. Then there exists such a V'', that $V'' \subset V' \cap V$ and $Cl(C' + V'') \subset C' + V$. Let $x, x + h \in V \cap \phi(x_0)$. Then $h \in (2V) \cap \phi(x_0)$. Let $\mu = P_{C'+V''}(h)$ where $\mu > 0$, then from (iv) follows that $R(x, h) = R(x, \mu(\mu^{-1}h) \in \mu C \subset \frac{1}{2}\mu C' \subset \frac{1}{2}\mu(C' + V'')$.

Since $\mu^{-1}h \in Cl(C' + V'') \cap \phi(x_0) \subset (C' + V) \cap \phi(x_0)$.

In this case $P_{C'+V''}(R(x, h)) \leq \frac{1}{2}\mu = \frac{1}{2}P_{C'+V''}(h)$, hence

$P_{C'+V''}(f(x + h) - f(x)) = P_{C'+V''}(h + R(x, h)) \geq \frac{1}{2}P_{C'+V''}(h)$……………..…………………...… (v)

From convexity and balancedness of the set $C' + V''$, it follows that if $\mu = P_{C'+V''}(h) = 0$, then (iv) is also fulfilled. From boundedness of C' and arbitrariness of V' and the definition of induced uniform structure on $H = V \cap \phi(x_0)$ and (v) follows that $f|_H: H \longrightarrow f(H)$ is bijective and uniform continuity of inverse mapping. ■

4.6.4 Corollary

If for any point x_0 in a topological manifold $\mathcal{W}$ on a cofinite space and any neighborhoods $U(x_0)$ and $V(x_0)$ of x_0 and almost continuous functions ϕ and ψ are such that they map $\mathcal{W}$ to cofinite spaces $\phi\big(U(x_0) \cap V(x_0)\big)$ and $\psi\big(U(x_0) \cap V(x_0)\big)$. Then $\exists$ a bijective almost continuous function $h: \phi(U(x_0) \cap V(x_0)) \longrightarrow \psi(U(x_0) \cap V(x_0))$.

Proof

Assuming that ϕ and ψ satisfy condition 1 of definition 4.3, then definition 4.6.1 together with theorems 4.6.2 and 4.6.3 proves corollary 4.6.4. ■

4.7 Application of manifolds

Manifolds are topological space that can be equipped with differentiable structure to become differentiable manifolds. This makes manifold to be applicable in many areas some of which are indicated below.

4.7.1 Survey

A sphere is a manifold and the earth is spherical hence it is a manifold that can be denoted by M. Examples of manifolds start with open domains in the Euclidean space $\mathbb{R}^n$. An important generalization of an open subset of $\mathbb{R}^n$ is that of an n - dimensional manifold. Such an object is obtained by suitably gluing open subsets $\{U_\alpha\}_{\alpha\in I}$ of $\mathbb{R}^n$ by smooth transition maps. Surveyors exploring neighboring regions would get maps such that some points of map U would appear in map V. These maps linked back to the manifold M using continuous functions such as ϕ_α and ϕ_β would overlap showing the points in common, that is the continuous functions ϕ_α and ϕ_β maps part of spherical manifold M to a flat surface. To consolidate these maps to a uniform scale, a continuous bijective function $\psi: U \longrightarrow V$ from map U to map V would be obtained. This would be done such that the transition between the maps is smooth.

4.7.2 Physics

a) Mechanics: we can define a differentiable manifold as a manifold together with a differentiable structure. A differentiable manifold is a space which looks locally like the Euclidean space but which globally may not. The theory of differentiable manifolds extends the ideas of calculus and analysis on $\mathbb{R}^n$ to these non-Euclidean spaces. From the differentiable manifold we can define tangent bundle and cotangent bundle. The tangent bundle is defined widely in Lagrangian formalism and is typically called the state space. It also describes the motion of objects in all classical mechanics scenarios. Differentiable manifolds naturally appear in various applications as configuration space in mechanics such as Hamiltonian mechanics. They are arguably the most general objects on which calculus can be developed. On the other hand differentiable manifolds provides for calculus a powerful invariant geometric language which is used in almost all areas of mathematics and its applications in vector fields, differential forms, integration on manifolds and de Rham cohomology. We consider a mechanical system whose configuration space is a differentiable manifold. According to Newton’s laws, the configuration of a system at some instant is not enough to determine the configuration at some other instant; we need also to know the momenta of the system at some instant to determine the evolution of the system. This momentum corresponds to a covector at a point p of the differentiable manifold M that represents the configuration of the system at that instant; therefore the cotangent bundle T^*M which we call the phase space determines the state of the system since it encodes the positions and the momenta of the objects.

b) Mesh generation: manifolds and differential geometry are frequently used in theoretical expositions of electromagnetics. The domain of a boundary value problem is covered with a coordinate system when each point of the domain is labeled with real numbers. A customary way to model the domain of a boundary value problem is to choose a particular coordinate system and use a subset of it as the domain hence the need of arithmetic to solve a boundary value problem. Manifolds on the other hand, reflect the principle that identification of points of the domain with coordinates is somewhat arbitrary. That is manifolds are point sets that can be represented with coordinate systems, emphasis being on the existence of coordinate systems, not on any particular coordinate system. Thus the primary object is the point set and the coordinate systems. In manifolds the coordinates are deliberately not bound to each other by distances. Mesh generation is an important step in numerical solution of a quasi-static electromagnetic boundary value problem with finite element (Raumonen *et al*., 2008). A boundary value problem to model electromagnetic phenomena is a systematization of a body of observations. The boundary value problem is posed to govern fields defined over a domain, a point set denoted by M. The points of M correspond to the points distinguishable by measurements with a rigid reference object. For practical purposes, M must be parameterized, that is covered with coordinate systems making M be locally Euclidean. From M to the coordinate spaces, there are continuous functions mappings called charts with continuous inverses. This allows charts to be local coordinate systems. If the charts are differentiable then we get differentiable manifolds. A collection of admissible charts defining the same differentiability of functions can be constructed and this allows differentiability not to depend on the choice of chart. The change of chart maps between any two admissible charts must be appropriately differentiable. Charts of this kind give an equivalent class called a differentiable structure. Finally such a differentiable manifold can be defined and this gives a possibility to pose boundary value problems.

CHAPTER FIVE
CONCLUSIONS AND RECOMMENDATIONS

This section gives a summary of the findings of this study which has looked at invariance of some topological properties with respect to almost continuous functions from a T_1 – space with cofinite topology (cofinite space) to the Euclidean R^n space. It also gives some of the recommendations for further research in terms of invariance and on the modeled space.

5.1 Conclusions

Some of the topological properties such as separability, compactness, sequential compactness, limit point compactness and pseudocompactness have been shown to be invariant of almost continuous functions from cofinite space to the Euclidean space $\mathbb{R}^n$. If some topological properties can be described in cofinite space but not in the Euclidean space $\mathbb{R}^n$ and are invariant from cofinite space to the Euclidean space $\mathbb{R}^n$, then through invariance these properties can be studied in cofinite space then inferred to the Euclidean space $\mathbb{R}^n$. This knowledge of invariance therefore is important since it can be used to study some of the topological properties which might not easily be studied in either of the spaces. *Pseudoderivatives* on almost continuous functions have also been defined and some of its properties have been stated based on the sum rule and the product rule. In the classical derivatives, these rules are valid and similarity can be realized with those stated for *pseudoderivatives* on almost continuous functions. Manifolds have been modeled on the cofinite space using almost continuous functions. This has been achieved using inverse function theorem as a bijection between the two cofinite spaces.

5.2 Recommendations

This thesis has only looked at invariance of some topological properties from the cofinite space to the Euclidean space $\mathbb{R}^n$. One can as well broaden up and look at other forms of compactness and other topological properties like connectedness and its other forms and investigate their invariance from the cofinite space to the Euclidean space $\mathbb{R}^n$. *Pseudoderivative* on almost continuous functions which has been defined has only been used to facilitate modeling. One can as well use it to solve problems in the class of T_1 – spaces. On the modeled space one can do more analysis by developing some of the properties studied in the manifold modeled on Hausdorff space.

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APPENDIX

Derivative

Let f be a function defined in an open neighborhood U_0 of a point $x \in \mathbb{R}$. The function f is called differentiable at x if there exists the limit $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ called the derivative of f at x. It follows that if f is differentiable at x, then it must be continuous at x.

Properties of Derivatives

a) Linearity

This property consists of two parts:

i. The derivative of a constant times a function is the constant times the derivative of the function, that is $\frac{d}{dx}[Cf(x)] = C\frac{d}{dx} f(x)$

Proof

Let $g(x) = Cf(x)$. Then $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$

$$= \lim_{h\to 0} \frac{Cf(x+h) - Cf(x)}{h}$$

$$= \lim_{h\to 0} C\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$= C \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= Cf'(x)$$

ii. The derivative of a sum of functions is the sum of the derivatives. If f and g are both differentiable, then $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

Proof

Let $F(x) = f(x) + g(x)$. Then $F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$

$$= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= f'(x)+g'(x)$$

Note: The sum rule can be extended to the sum of any number of functions. For instance using the property (ii) above we get $(f+g+h)' = [(f+g)+h]' = (f+g)' + h' = f' + g' + h'$. Usually, the above two properties can combined into one, $[af(x)+bg(x)]' = af'(x)+bg'(x)$ where a and b are constants, whereas $f(x)$ and $g(x)$ are functions. The expression in the brackets on the left hand side is referred to as *linear combination*. Thus the linearity can be stated as ' Derivative of a linear combination is equals to the linear combination of derivatives'

b) The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function. If f and g are both differentiable, then $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$

Proof

Let $h(x) = f(x)g(x)$, then

$$h'(x) = \lim_{h\to 0}\frac{h(x+h)-h(x)}{h}$$

$$= [f(x)g(x)]'$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]+[f(x+h)-f(x)]g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)]}{h}+\lim_{h\to 0}\frac{[f(x+h)-f(x)]g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{[g(x+h)-g(x)]}{h}+g(x)\lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h}$$

$$= f(x)\lim_{h\to 0}\frac{[g(x+h)-g(x)]}{h}+g(x)\lim_{h\to 0}\frac{[f(x+h)-f(x)]}{h}$$

$$= f(x)g'(x)+g(x)f'(x)$$

Fréchet derivative

Let $\{U,\|\cdot\|_U\}$ and $\{V,\|\cdot\|_V\}$ denote the real Banach spaces and $\mathfrak{A}$ be an open subset of U. A mapping $F:\mathfrak{A}\subset U\longrightarrow V$ is said to Fréchet differentiable at $u\in\mathfrak{A}$ if there exists an operator $A\in L(U,V)$ and a mapping $r(u,\cdot):U\longrightarrow V$ with the following properties; for all $h\in U$ such that $u+h\in\mathfrak{A}$, we have $F(u+h)=F(u)+Ah+r(u,h)$ where the so called remainder r satisfies the condition $\frac{\|r(u,h)\|_V}{\|h\|_U}\to 0$ as $\|h\|_U\longrightarrow 0$. The operator A is then called the *Fréchet derivative* of F at u and we write $A=F'(u)$. If A is Fréchet differentiable at every point $u\in\mathfrak{A}$, then A is said to be Fréchet differentiable in $\mathfrak{A}$. Since $\mathfrak{A}$ is an open set, we have $u+h\in\mathfrak{A}$ for all $h\in U$ with sufficiently small norm. Hence the relation to be satisfied by the remainder $r(u,h)$ is meaningful at least for all $h\in U$ from a small ball about the origin.

In elementary calculus the derivative at a point x is the local linear approximation of the given function in the neighborhood of x. Similarly the Fréchet derivative can be interpreted as the best local linear approximation. Therefore the existence of classical derivative at a point x implies the existence of Fréchet derivative at a point x

Definition

Let M and N be any two sets. A function $f:M\longrightarrow N$ is a rule that assigns to every element $m\in M$ a unique element $n\in N$. We then call n the image of m under f. The set M is called the *domain* of f while N is called the *codomain* of f. The set $f(M)$ of elements of N which are images of members of M under f is called the *range* of f.

Continuity of a function

Let X and Y be the topological spaces. A function $f:X\longrightarrow Y$ is said to be continuous if each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X. Recall that $f^{-1}(V)$ is a set of all points x of X for which $f(x)\in V$; it is empty if V does not intersect the image set $f(x)$ of f. Continuity of a function depends not only upon the function f itself, but also on the topologies specified for its domain and range.

Local criteria for continuity

A map $f: X \longrightarrow Y$ between topological spaces is continuous if and only if each point of X has a neighborhood on which the restriction of f is continuous.

Proof

If f is continuous, we may simply take each neighborhood to be X itself. Conversely, suppose f is continuous in a neighborhood of each point, and let $U \subset Y$ be any open set, we have to show that $f^{-1}(U)$ is open. Any point $x \in f^{-1}(U)$ has a neighborhood V_x on which f is continuous. Continuity of $f|_{V_x}$ implies in particular that $\left(f|_{V_x}\right)^{-1}(U)$ is open in V_x and therefore also open in X. Unwinding the definition, we see that $\left(f|_{V_x}\right)^{-1}(U) = \{x \in V_x: f(x) \in U\} = f^{-1}(U) \cap V_x$ which contains x and is contained in $f^{-1}(U)$. Since $f^{-1}(U)$ is the union of all such open sets as x ranges over $f^{-1}(U)$, it follows that $f^{-1}(U)$ is open as desired.

Definitions: Let f be a function from a set X into a set Y:

1) The function f is said to be *one-to one* or *injective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for $x_1, x_2 \in X$.
2) The function f is said to be *onto* or *surjective* if each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.
3) The function f is said to be *bijective* if it is both *one-to-one* and *onto*.

Definition: consider topological spaces X, Y and a function $f: X \longrightarrow Y$. Then f is a connectivity function if for every connected subset E of X, the restriction of f to E has the connected graph.

In T_1- spaces some properties of continuous functions are also shared by almost continuous functions. These have been illustrated by the following propositions;

Proposition 1: Let X, Y be T_1- spaces. Let $f: X \longrightarrow Y$ be an almost continuous function and $g: X \longrightarrow Y$ be the continuous function whose graph is contained in every neighborhood of $G(f)$. If f is injective so is g.

Proof

Suppose g is not injective. Then there exists $y_0 \in Y$ and $x, x' \in X$ such that (x, y_0), $(x', y_0) \in G(g)$. Clearly since f is one-to-one either $(x, y_0) \notin G(f)$ or $(x', y_0) \notin G(f)$.

Without loss of generality, let $(x', y_0) \notin G(f)$. Then $G(f) \subseteq X \times Y \setminus \{(x', y_0)\}$. Since $X \times Y$ is a $T_1 -$space, $X \times Y \setminus \{(x', y_0)\}$ is an open neighborhood of $G(f)$ in $X \times Y$. But $G(g) \subsetneq X \times Y \setminus \{(x', y_0)\}$. This contradicts the hypothesis that $G(g)$ is contained in every neighborhood of $G(f)$. Thus the supposition that g is not a one-to-one function has led to a contradiction and can therefore not hold. Similarly, if f is a bijection (that is a one-to-one and onto function) g is also one-to-one and we have that the Cardinality of $g(X) =$ cardinality of $f(X) =$ cardinality of Y; that is $g(X) = Y$. ■

Proposition 2: Let $f: X \longrightarrow Y$ be an almost continuous function from a T_1 – space X into a T_1 – space Y. Let g be the continuous function whose graph is contained in every neighborhood of $G(f)$. If f is a bijection so is g.

Partition of unity

A partition of unity on a differentiable manifold M is a collection $\{\psi_i\}_{i \in I}$ of smooth functions such that:

1. $\psi_i(p) \geq 0$ for every $p \in M$
2. the collection of supports $\{supp\ \psi_i : i \in I\}$ is locally finite
3. $\sum_{i \in I} \psi_i(p) = 1$ for every $p \in M$

The support of a smooth function is the closure of the set in its domain where it takes on nonzero values. The support of f is denoted as $supp\ f = \overline{\{p \in M : f(p) \neq 0\}}$. Given a collection $\beta = \{U_\alpha : \alpha \in A\}$ of subsets of M, we say that:

a) β is locally finite if for all $p \in M$, there exists a neighborhood $p \in O \subset M$ such that $O \cap U_\alpha \neq 0$ for only a finite number of $\alpha \in A$.
b) β is a cover of M if $\bigcup_{\alpha \in A} U_\alpha = M$.
c) $\xi = \{\mathfrak{U}_\phi : \phi \in B\}$ is a subcover if $\xi \in B$ and ξ still covers M.
d) $\beta' = \{V_i : i \in I\}$ is a refinement of a cover β if there exists $\alpha = \alpha(i) \in A$ such that $V_i \subset U_\alpha$

A partition of unity $\{\psi_i : i \in I\}$ is called subordinated to a cover $\{U_\alpha : \alpha \in A\}$ of M if for each $i \in I$ there exists $\alpha \in A$ such that the $supp\ \psi_i \subset U_\alpha$.

Bump function

Let M be a smooth manifold. For any closed set $A \subset M$ and any open set U containing A, there exists a smooth function $\varphi: M \longrightarrow \mathbb{R}$ such that $\varphi \equiv 1$ on A and the $supp\ \varphi \subset U$. Let $U_0 = U$ and $U_1 = M - A$ and let $\{\varphi_0, \varphi_1\}$ be a partition of unity subordinate to the open cover $\{U_0, U_1\}$. Because $\varphi_1 \equiv 0$ on A and therefore $\varphi_0 = \sum_i \varphi_i = 1$ there, the function φ_0 has the required properties. Any function with the properties described above is called bump function for A supported in U.

Existence of partition of unity

If M is a smooth manifold and $\zeta = \left(X_\mu\right)_{\mu \in \mathrm{A}}$ is any open cover of M, there exists a partition of unity subordinate to ζ. Let $\{W_i\}$ be a regular refinement of ζ. For each i, let $\psi_i: W_i \longrightarrow B_3(o)$ be a diffeomorphism whose existence is guaranteed by the definition of a regular cover and let $\mathfrak{U}_i = {\psi_i}^{-1}\left(B_1(0)\right)$ and $B_i = {\psi_i}^{-1}\left(B_2(0)\right)$. For each i, define a function $f_i: M \longrightarrow \mathbb{R}$ by $f_i = \begin{cases} H \circ \psi_i \ on \ \ W_i \\ 0 \ on \ \ M - \overline{V_i} \end{cases}$ where $H: \mathbb{R}^n \longrightarrow \mathbb{R}$ is the bump function. On the set $W_i - \overline{V_i}$ where the two definitions overlap, both definitions yields the zero functions so f_i is well defined and smooth and the $supp\ f_i \subset W_i$. Define a new function $g_i: M \longrightarrow \mathbb{R}$ by $g_i(x) = \dfrac{f_i(x)}{\sum_j f_i(x)}$. Because of local finiteness of the cover $\{W_i\}$, the sum in the denominator has only finitely many nonzero terms in the neighborhood of each point and thus defines a smooth function. Because $f_i \equiv 1$ on $\mathfrak{U}_i$ and every point of M is in some $\mathfrak{U}_i$, the denominator is always positive, so g_i is a smooth function on M. It is immediate from the definition that $0 \le g_i \le 1$ and $\sum_i g_i \equiv 1$. Re-indexing the functions so that they are indexed by the same set A as the open cover. For each i, there is some index $a(i) \in A$ such that $W_i \subset X_{a(i)}$. For each $\alpha \in A$, define $\psi_\alpha: M \longrightarrow \mathbb{R}$ by $\psi_\alpha = \sum_{i:a(i)=\alpha} g_i$. Each ψ_α is smooth and satisfies $0 \le \psi_\alpha \le 1$ and the $supp\ \psi_\alpha \subset X_\alpha$. Moreover the set of support $\{supp\ \psi_\alpha\}_{\alpha \in A}$ is still locally finite and the $\sum_\alpha \psi_\alpha \equiv \sum_i g_i \equiv 1$, hence the desired partition of unity.

Urysohn's lemma: Let X be a normal space. The closed subsets of X can be separated by functions. For $A, B \subseteq X$ closed and disjoint, there is a continuous function $f: X \longrightarrow [0,1]$ satisfying $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. Sometimes such a function is called Urysohn function for A and B.